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ANALYTICITY AND ASYMPTOTIC PROPERTIES OF THE MAXWELL OPERATOR'S DISPERSION CURVES

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The fibers of the elliptic Maxwell operator M in an infinite cylindrical wave guide are not of type (B) in Kato's sense. In spite of this phenomenon, which apparently does not appear with the other operators of mathematical physics, we prove that the dispersion curves $(\lambda_n)_{n\geq 1}$ of M are real analytic functions. Nevertheless, it is not yet known if they are monotone. Therefore, we define the thresholds of M as the stationary points of $p \mapsto \lambda_n(p)$, for any $n \geq 1$. This approach generalizes the common definition of the thresholds used in acoustics. Next, the asymptotic behavior of the dispersion curves with respect to parameters p and n allow us to deduce a limiting absorption principle for M, which remains valid at the thresholds.

1. Introduction

In Refs. 13 and 14, we study the spectral problems of electromagnetic wave propagation in a three-dimensional layered medium. In this paper, we examine the more general situation described in Ref. 16, of an infinite cylindrical wave guide, whose cross-section is bounded.

More precisely, Ω_T is a bounded, connected and simply connected open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary Γ_T. The infinite cylinder $\Gamma = \Gamma_T \times \mathbb{R}$ represents a perfectly conducting wave guide, and the dielectric permittivity ε together with the magnetic permeability μ of the continuous isotropic propagation medium $\Omega = \Omega_T \times \mathbb{R}$, are real measurable and strictly positive functions that depend only on the cross variables $x_T = (x_1, x_2) \in \Omega_T$. Next, n_Γ denoting the unit outward normal to Γ and $x = (x_T, x_3)$ being the usual point of Ω, the electric field u propagating in the medium Ω satisfies, according to the Maxwell equations detailed in Ref. 4,

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$$\begin{cases} \partial_t^2 u(x,t) + \mathcal{M}u(x,t) = 0, \ (x,t) \in \Omega \times \mathbb{R}_+^*\,, \\ \mathrm{div}(\varepsilon u)(x,t) = 0, \ (x,t) \in \Omega \times \mathbb{R}_+^*\,, \\ u(\sigma,t) \wedge n_\Gamma(\sigma) = 0, \ (\sigma,t) \in \Gamma \times \mathbb{R}_+^*\,, \end{cases}$$

where $\mathcal{M}u = \varepsilon^{-1}\,\mathrm{curl}(\mu^{-1}\,\mathrm{curl}\ u)$. In the following, we assume in addition that $\varepsilon^{\pm 1}$ and $\mu^{\pm 1}$ belong to $L^\infty(\Omega_T)$. Note that under these assumptions, $\mathcal{M}$ does not reduce to the three-dimensional Laplace operator.

The differential second-order operator $\mathcal{M}$ has an elliptic self-adjoint realization M in the Hilbertian space

$$H_\varepsilon = \{u \in L^2(\Omega)^3, \ \mathrm{div}(\varepsilon u) = 0\}$$

endowed with the $L^2(\Omega; \varepsilon\, dx_T\, dx_3)^3$ scalar product. This operator is called second-order Maxwell operator or elliptic Maxwell operator. It is defined by

$$\begin{cases} D(M) = \{u \in H(\mathrm{curl};\Omega) \cap H_\varepsilon, \ \gamma_\tau u = 0, \ \mu^{-1}\,\mathrm{curl}\ u \in H(\mathrm{curl};\Omega)\} \\ \forall\, u \in D(M), \ Mu = \mathcal{M}u\,, \end{cases}$$

where $H(\mathrm{curl};\Omega)$ denotes the space $\{u \in L^2(\Omega)^3, \ \mathrm{curl}\ u \in L^2(\Omega)^3\}$ equipped with the norm $\|u\|_{H(\mathrm{curl};\Omega)} = (\|u\|^2_{L^2(\Omega)^3} + \|\mathrm{curl}\ u\|^2_{L^2(\Omega)^3})^{1/2}$, and γ_τ is the unique continuous linear mapping from $H(\mathrm{curl};\Omega)$ into $H^{-1/2}(\Gamma)^3$ such that $\gamma_\tau u = u \wedge n_\Gamma$ for any $u \in C_0^\infty(\bar{\Omega})^3$. In fact, the self-adjointness of M arises from the following "Green-formula", which is valid for u in $H(\mathrm{curl};\Omega)$ and v in $H^1(\Omega)^3$,

$$(\mathrm{curl}\ u, v)_{L^2(\Omega)^3} = (u, \mathrm{curl}\ v)_{L^2(\Omega)^3} + \langle \gamma_\tau u, \gamma_0 v\rangle_{H^{-1/2}(\Gamma)^3, H^{1/2}(\Gamma)^3}\,, \tag{1.1}$$

where γ_0 denotes the classical trace function from $H^1(\Omega)^3$ onto $H^{1/2}(\Gamma)^3$.

Thus, the spectrum of M, $\sigma(M)$, is a real set, and the resolvent operator $z \mapsto R_M(z) = (M - z)^{-1}$ is also an analytic function from $\mathbb{C}\backslash\mathbb{R}$ into $B(H_\varepsilon)$. But, it is well known that $\lim_{z\to\tau} \|R_M(z)\|_{B(H_\varepsilon)} = +\infty$ when τ belongs to $\sigma(M)$. We will prove in Theorems 3.1 and 3.2 that $z \mapsto R_M(z)$ can be extended (in a suitable weighted L^2-topology) to the lower or upper half plane $\overline{\mathbb{C}^\pm}$ in a locally Hölder continuous function, $R_M^\pm$. This continuous extension is called "limiting absorption principle". The local Hölder continuity of the extended resolvent operator allows the study of the time-asymptotic behavior of operator $t \mapsto e^{iM^{1/2}t}$. More precisely, this property is very useful for showing in the same way as in Ref. 9 for the acoustic propagator, that the solution u of the Cauchy problem

$$\partial_t^2 u(x,t) + Mu(x,t) = e^{-it\sqrt{\omega}} f(x), \quad \omega \in \mathbb{R}_+^*\,,$$

for a convenient data function f, with zero initial conditions $u(x,0) = \partial_t u(x,0) = 0$, behaves like

$$u(x,t) = e^{-it\sqrt{\omega}} R_M^\pm(\omega) f(x) + o(1)$$

as t goes to $\pm\infty$. Then, as it is explained in Ref. 12, the knowledge of this behavior allows us to prove the existence and completeness of the generalized wave operator, which is fundamental for the scattering theory.

The proof of the limiting absorption principle for M is given in Sec. 3. It requires an adapted spectral representation of M that will be built preliminarily in Sec. 2.

Similar spectral problems for acoustic operators have already been studied by several authors in Ref. 1 or 3. Under suitable assumptions on the behavior of sound speed, it is proved that the dispersion curves are strictly monotone on $\mathbb{R}_+^*$, so they can deduce a limiting absorption principle for the acoustic propagator. In fact, this very particular behavior is not essential for the proof of a limiting absorption principle. Indeed, by only using the analyticity together with the following asymptotic properties

$$\forall\ n \geq 1, \lim_{|p|\to+\infty} \lambda_n(p) = +\infty \quad \text{and} \quad \lim_{n\to\infty}\left(\inf_{p\in\mathbb{R}} \lambda_n(p)\right) = +\infty\,,$$

of the dispersion curves $(\lambda_n)_{n\geq 1}$ of M, we prove in the following that R_M can be extended in a locally Hölder continuous function on the real axis. Note that the stationary points of each dispersion curve above may eventually be a discrete set of $\mathbb{R}$. Furthermore, contrary to the acoustic or elastic case, the analytic properties of $(\lambda_n)_{n\geq 1}$ do not arise directly from Kato's theory detailed in Ref. 8 because the holomorphic family associated to the fibers of M is neither of type (A) nor of type (B). Besides, it seems that this phenomenon also does not appear with the other classical operators of mathematical physics. Therefore, the proof in Sec. 2.3 of the analytic properties of any $p \mapsto \lambda_n(p)$, for $n \geq 1$, requires a preliminary study in Sec. 2.2 of the behavior of these dispersion curves in a neighborhood of 0.

2. Spectral Analysis of M

The spectral analysis of M is essentially based on the use of the partial Fourier transform with respect to the infinite direction x_3. Indeed, as ε depends only on the cross variable $x_T \in \Omega_T$, the mapping

$$\mathcal{F}_{x_3} : u \mapsto \hat{u}(x_T, p) = (2\pi)^{-1/2} \lim_{X\to+\infty} \int_{-X}^{X} u(x_T, x_3) e^{-ipx_3} dx_3\,,$$

defines a unitary transform from $L^2(\Omega; \varepsilon\, dx_T\, dx_3)^3$ onto $L^2(\Omega; \varepsilon\, dx_T\, dp)^3$.

2.1. *Definition of the reduced operators of M*

Set $\mathrm{curl}_T\, u_T = \partial_{x_1} u_2 - \partial_{x_2} u_1$ for any $u_T = (u_1, u_2)$ and let $n_{\Gamma_T} = (n_1, n_2)$ be the unit outward normal to Γ_T. Then, according to Ref. 5, the mapping $\gamma_\tau^T u_T = u_1 n_2 - u_2 n_1$, which is defined on $\overline{C_0^\infty(\Omega_T)}^2$, extends uniquely in a continuous linear application γ_τ^T from $H(\mathrm{curl}_T; \Omega_T) = \{v \in L^2(\Omega_T)^2,\ \mathrm{curl}_T\, v \in L^2(\Omega_T)\}$ endowed with its usual topology, onto $H^{-1/2}(\Gamma_T)$. Next, for any $u \in H(\mathrm{curl}; \Omega)$, a simple computation gives

$$\mathcal{F}_{x_3}(\mathrm{curl}\ u)(x_T, p) = \mathrm{curl}_p\, \hat{u}(x_T, p) \quad \text{a.e. } (x_T, p) \in \Omega\,,$$

with $\operatorname{curl}_p u = (\partial_{x_2} u_3 + ipu_2, -ipu_1 - \partial_{x_1} u_3, \operatorname{curl}_T u_T)$. For each real p, we deduce from (1.1) that

$$\begin{aligned}&(\operatorname{curl}_p u, v)_{L^2(\Omega_T)^3} - (u, \operatorname{curl}_p v)_{L^2(\Omega_T)^3} \\ &\quad = -\left\langle \gamma_\tau^T u_T, \gamma_0^T v_3 \right\rangle_{H^{-1/2}(\Gamma_T), H^{1/2}(\Gamma_T)} + \left\langle \gamma_0^T u_3, \gamma_\tau^T v_T \right\rangle_{H^{1/2}(\Gamma_T), H^{-1/2}(\Gamma_T)}, \end{aligned} \tag{2.2}$$

for any u and v in $H(\operatorname{curl}_T; \Omega_T) \times H^1(\Omega_T)$, where γ_0^T is the trace function from $H^1(\Omega_T)$ onto $H^{1/2}(\Gamma_T)$. Thus, for any $u \in H(\operatorname{curl}; \Omega)$, the "Green formula" (2.2) warrants:

$$(\gamma_\tau u = 0) \Leftrightarrow (\gamma_\tau^T \hat{u}_T(\cdot, p) = 0 \quad \text{and} \quad \gamma_0^T \hat{u}_3(\cdot, p) = 0 \quad \text{a.e. } p \in \mathbb{R}).$$

This leads to define the "reduced operators" of M, called M_p, $p \in \mathbb{R}$, as

$$\begin{cases} D(M_p) = \{u \in W_p, \mu^{-1} \operatorname{curl}_p u \in H(\operatorname{curl}_T; \Omega_T) \times H^1(\Omega_T)\} \\ \forall\, u \in D(M_p),\ M_p u = \varepsilon^{-1} \operatorname{curl}_p(\mu^{-1} \operatorname{curl}_p u), \end{cases}$$

where $H_{\varepsilon,p}$ denotes the space $\{u \in L^2(\Omega_T)^3,\ \partial_{x_1}(\varepsilon u_1) + \partial_{x_2}(\varepsilon u_2) - ip\varepsilon u_3 = 0\}$ endowed with the $L^2(\Omega_T; \varepsilon\, dx_T)^3$ scalar product, and $W_p = H_{\varepsilon,p} \cap (\ker \gamma_\tau^T \times H_0^1(\Omega_T))$. Thus, operator M is unitarily equivalent to the direct integral along $\mathbb{R}$ whose fibers are M_p:

$$M = \mathcal{F}_{x_3}^* \left(\int_{p \in \mathbb{R}}^{\oplus} M_p\, dp \right) \mathcal{F}_{x_3}. \tag{2.3}$$

Now, fix p in $\mathbb{R}$. The Green formula (2.2) shows that M_p is a symmetric operator, and, for any $u \in D(M_p)$, we have in addition

$$(M_p u, u)_{H_{\varepsilon,p}} = \|\mu^{-1/2} \operatorname{curl}_p u\|_{L^2(\Omega_T)^3}^2 \geq c_{\min}^2 \|\operatorname{curl}_p u\|_{H_{\varepsilon,p}}^2. \tag{2.4}$$

Next, a simple computation gives

$$\begin{aligned}\|\operatorname{curl}_p u\|_{H_{\varepsilon,p}}^2 &= \|\varepsilon^{1/2} \nabla_T u_3\|_{L^2(\Omega_T)^2}^2 + \|\varepsilon^{1/2} \operatorname{curl}_T u_T\|_{L^2(\Omega_T)}^2 \\ &\quad - 2p \operatorname{Im}((\varepsilon u_T, \nabla_T u_3)_{L^2(\Omega_T)^2}) + p^2 \|\varepsilon^{1/2} u_T\|_{L^2(\Omega_T)^2}, \end{aligned} \tag{2.5}$$

with $\nabla_T u_3 = (\partial_{x_1} u_3, \partial_{x_2} u_3)$. But, as $u \in H_{\varepsilon,p}$, we have $(\varepsilon u_T, \nabla_T u_3)_{L^2(\Omega_T)^2} = -ip\|\varepsilon^{1/2} u_3\|_{L^2(\Omega_T)}^2$, so we finally get

$$(M_p u, u)_{H_{\varepsilon,p}} \geq c_{\min}^2 p^2 \|u\|_{H_{\varepsilon,p}}^2 \quad \text{with} \quad c_{\min}^2 = \inf_{x_T \in \Omega_T} (\varepsilon\mu)^{-1}(x_T).$$

This inequality shows that

$$M_p \text{ is bounded from below by } c_{\min}^2 p^2 \text{ for any } p \in \mathbb{R}. \tag{2.6}$$

2.2. *Dispersion curves of M*

C. Weber proved in Ref. 17 that the imbedding $W_p \hookrightarrow H_{\varepsilon,p}$ is compact for any $p \in \mathbb{R}$. Thus, M_p, $p \in \mathbb{R}$, is a self-adjoint operator with compact resolvent in $H_{\varepsilon,p}$. Its spectrum, $\sigma(M_p)$, is also discrete and we have

$$\sigma(M_p) = \{\Lambda_n(p), n \geq 1\}\,, \tag{2.7}$$

with $\Lambda_n(p) \leq \Lambda_{n+1}(p)$ for any $n \in \mathbb{N}^*$. According to (2.6), the dispersion curves $(\Lambda_n)_{n\geq 1}$ satisfy in addition:

$$\forall\, n \geq 1,\ \forall\, p \in \mathbb{R},\ \Lambda_n(p) \geq c_{\min}^2 p^2\,. \tag{2.8}$$

Moreover, these curves cannot reach 0 because

$$\Lambda_1(0) > 0\,. \tag{2.9}$$

Indeed, assume for the moment that $\Lambda_1(0) = 0$. Then, according to (2.5), any eigenvector $\varphi = (\varphi_T, \varphi_3)$ of M_0 associated to $\Lambda_1(0)$ satisfies $\nabla_T \varphi_3 = 0$ and $\operatorname{curl}_T \varphi_T = 0$ in the same time. Furthermore, φ_3 belongs to $H_0^1(\Omega_T)$, so $\varphi_3 = 0$. Next, as $\varphi_T \in \ker \gamma_\tau^T$, it derives from the Poincaré lemma detailed in Ref. 7, that $\varphi_T = \nabla_T f$ for some $f \in H_0^1(\Omega_T)$. But φ is in W_0, so φ_T belongs to the space $H_{\varepsilon,T}$ defined by

$$H_{\varepsilon,T} = \{u_T = (u_1, u_2) \in L^2(\Omega_T)^2,\ \operatorname{div}_T(\varepsilon u_T) = \partial_{x_1}(\varepsilon u_1) + \partial_{x_2}(\varepsilon u_2) = 0\}\,.$$

Therefore, the following Weyl–Hodge orthogonal decomposition given in Refs. 10 and 15,

$$L^2(\Omega_T; \varepsilon\, dx_T)^2 = H_{\varepsilon,T} \oplus \nabla_T H_0^1(\Omega_T) \tag{2.10}$$

asserts that $\varphi_T = 0$. This finally shows that $\Lambda_1(0) > 0$.

Now, using (2.9), we will prove the following essential property for the proof of a limiting absorption principle for M.

Lemma 2.1. *The dispersion curves cannot be arbitrarily near* 0:

$$\inf_{(n,p)\in\mathbb{N}^*\times\mathbb{R}} \Lambda_n(p) > 0\,.$$

Proof. Fix p in $\mathbb{R}$ and $u = (u_T, u_3)$ in $D(M_p)$. Then, the decomposition (2.10) assigns $u_T = v_T + \nabla_T f$ with $v_T \in H_{\varepsilon,T}$ and $f \in H_0^1(\Omega_T)$. As $u \in H_{\varepsilon,p}$ and $v_T \in H_{\varepsilon,T}$, we also have

$$(\operatorname{div}_T(\varepsilon(u_T - v_T)), f)_{L^2(\Omega_T)} = -\|\varepsilon^{1/2}(u_T - v_T)\|^2_{L^2(\Omega_T)^2} = ip(\varepsilon u_3, f)_{L^2(\Omega_T)}\,.$$

Next, using the Cauchy–Schwarz and Poincaré inequalities, we get

$$\|\varepsilon^{1/2}(u_T - v_T)\|^2_{L^2(\Omega_T)^2} \leq C|p|\|\varepsilon^{1/2}u_3\|_{L^2(\Omega_T)}\|\varepsilon^{1/2}\nabla_T f\|_{L^2(\Omega_T)^2}\,,$$

where C is a constant depends only on Ω_T. This immediately gives

$$\|\varepsilon^{1/2}(u_T - v_T)\|_{L^2(\Omega_T)^2} \leq C|p|\|\varepsilon^{1/2}u_3\|_{L^2(\Omega_T)} . \tag{2.11}$$

Now, recall that the MaxMin principle warrants $\Lambda_1(0) = \inf_{w \in W_0, w \neq 0} \frac{(M_0 w, w)_{H_{\varepsilon,0}}}{\|w\|^2_{H_{\varepsilon,0}}}$. Thus, as v_T is in $\ker \gamma_\tau^T \cap H_{\varepsilon,T}$, the vector $v = (v_T, 0)$ belongs to W_0, so we get

$$\frac{(M_0 v, v)_{H_{\varepsilon,0}}}{\|v\|^2_{H_{\varepsilon,0}}} = \frac{\|\mu^{-1/2} \operatorname{curl}_T v_T\|^2_{L^2(\Omega_T)}}{\|v_T\|^2_{H_{\varepsilon,T}}} \geq \Lambda_1(0) ,$$

when $v_T \neq 0$. Therefore, by setting $c^2_{\max} = \sup_{x_T \in \Omega_T} (\varepsilon\mu)^{-1}(x_T)$ and $r = c^{-1}_{\max}\Lambda_1(0)^{1/2}$, it finally appears that

$$\|\varepsilon^{1/2} \operatorname{curl}_T v_T\|_{L^2(\Omega_T)} \geq r\|\varepsilon^{1/2}\, v_T\|_{L^2(\Omega_T)^2} ,$$

with, according to (2.9), $r > 0$. As $\operatorname{curl}_T u_T = \operatorname{curl}_T v_T$, we also derive from (2.11) that

$$\|\varepsilon^{1/2} \operatorname{curl}_T u_T\|_{L^2(\Omega_T)} \geq r(\|\varepsilon^{1/2}\, u_T\|_{L^2(\Omega_T)^2} - C|p|\|\varepsilon^{1/2}\, u_3\|_{L^2(\Omega_T)}) .$$

Thus, we have $\|\varepsilon^{1/2} \operatorname{curl}_T u_T\|^2_{L^2(\Omega_T)} \geq r^2(\|\varepsilon^{1/2}\, u_T\|^2_{L^2(\Omega_T)^2} - C|p|\|\varepsilon^{1/2}\, u\|^2_{L^2(\Omega_T)^3})$ for a sufficiently small p, and we deduce from (2.4) and (2.5) that

$$(M_p u, u)_{H_{\varepsilon,p}} \geq c^2_{\min}(K^2 - r^2 C|p|)\|\varepsilon^{1/2}\, u\|^2_{L^2(\Omega_T)^3} , \tag{2.12}$$

for some strictly positive constant K. At last, Lemma 2.1 follows immediately from (2.8) and (2.12). □

2.3. *Analyticity of the dispersion curves*

The equalities (2.3) and (2.7) involve that M is unitarily equivalent to the multiplication operator by $(\Lambda_n)_{n \geq 1}$ in a convenient space. Using the MaxMin principle, we show in Ref. 15 that any function $p \mapsto \Lambda_n(p)$, $n \geq 1$, is odd, continuous on $\mathbb{R}$ and a.e. differentiable on $\mathbb{R}^*$. However, the proof of this result requires so many worrying computations that we prefer to ignore them in this paper, although Lemma 2.1 is slightly more complicated to justify without any assumption of continuity of the dispersion curves. Moreover, in view of proving a limiting absorption principle for M, we need "analytic properties" and not only continuity for $p \mapsto \Lambda_n(p)$, $n \geq 1$. Thus, we will show that $(\Lambda_n)_{n \geq 1}$ can be rearranged in a family of analytic functions. To make this, we first generalize the definition of M_p to complex parameter p. In this case, the Green formula (2.2) becomes

$$\begin{aligned}(\operatorname{curl}_p u, v)_{L^2(\Omega_T)^3} &- (u, \operatorname{curl}_{\bar{p}} v)_{L^2(\Omega_T)^3} \\ &= \left\langle \gamma_\tau^T u_T, \gamma_0^T v_3 \right\rangle_{H^{-1/2}(\Gamma_T), H^{1/2}(\Gamma_T)} - \left\langle \gamma_0^T u_3, \gamma_\tau^T v_T \right\rangle_{H^{1/2}(\Gamma_T), H^{-1/2}(\Gamma_T)} ,\end{aligned} \tag{2.13}$$

and the sesquilinear form associated to M_p is also

$$m_p(u, v) = (\mu^{-1} \operatorname{curl}_p u, \operatorname{curl}_{\bar{p}} v)_{L^2(\Omega_T)^3} .$$

Its domain is $D(m_p) = W_p$, and, for any $u \in D(m_p)$, we get in the same way as with (2.5),

$$m_p(u,u) = \|\mu^{-1/2}\,\nabla_T u_3\|^2_{L^2(\Omega_T)^2} + \|\mu^{-1/2}\,\mathrm{curl}_T\, u_T\|^2_{L^2(\Omega_T)} - 2p\,\mathrm{Im}((\mu^{-1}u_T, \nabla_T u_3)_{L^2(\Omega_T)^2}) + p^2\|\mu^{-1/2}\,u_T\|^2_{L^2(\Omega_T)^2}\,. \quad (2.14)$$

Thus, for any fixed $u \in D(m_p)$, $p \mapsto m_p(u,u)$ is holomorphic. But, $D(m_p)$ is a subset of $H_{\varepsilon,p}$, so it depends on p. Therefore, the holomorphic family $\{M_p, p \in \mathbb{C}\}$ is not of type (B) in Kato's sense defined in VII.4.4 of Ref. 8, and we cannot conclude directly that the dispersion curves of M are real analytic functions as in acoustics (see Ref. 3) or in elasticity (see Ref. 2). However, we will prove with the next proposition that this result remains valid all the same.

Proposition 2.1. *There are two analytic families with respect to* $p \in \mathbb{R}$, $\{\lambda_n(p), n \in \mathbb{N}^*\}$ *and* $\{\varphi_n(p), n \in \mathbb{N}^*\}$, *such that, for any real* p:

(i) $\sigma(M_p) = \{\lambda_n(p),\ n \in \mathbb{N}^*\}$ *and* $\varphi_n(p)$, $n \in \mathbb{N}^*$, *is an eigenfunction of* M_p *associated to the eigenvalue* $\lambda_n(p)$,

(ii) $\{\varphi_n(p),\ n \in \mathbb{N}^*\}$ *is an orthonormal basis of* $H_{\varepsilon,p}$.

Proof. Let A denote the operator $M \oplus \underline{0}$ associated to the Weyl–Hodge orthogonal decomposition $L^2(\Omega; \varepsilon\, dx_T\, dx_3)^3 = H_\varepsilon \oplus \nabla H^1_0(\Omega)$ of Ref. 10. The operator A is self-adjoint in $L^2(\Omega; \varepsilon\, dx_T\, dx_3)^3$, and we immediately have $A = \mathcal{F}^*_{x_3}(\int^{\oplus}_{p\in\mathbb{R}} A_p\, dp)\mathcal{F}_{x_3}$, with

$$\begin{cases} D(A_p) = \{u \in \ker \gamma^T_\tau \times H^1_0(\Omega_T),\ \mu^{-1}\,\mathrm{curl}_p\, u \in H(\mathrm{curl}_T; \Omega_T) \times H^1(\Omega_T)\} \\ \forall\ u \in D(A_p),\ A_p u = \varepsilon^{-1}\,\mathrm{curl}_p(\mu^{-1}\,\mathrm{curl}_p\, u)\,, \end{cases}$$

for any $p \in \mathbb{R}$. This definition can be easily generalized to complex parameters p. According to (2.13), the sesquilinear form associated to A_p is defined by $a_p(u,v) = (\mu^{-1}\,\mathrm{curl}_p\, u, \mathrm{curl}_{\bar{p}}\, v)_{L^2(\Omega_T)^3}$ for any u and v in $D(a_p) = \ker \gamma^T_\tau \times H^1_0(\Omega_T)$. It is continuous on $D(a_p)^2$, and we have:

$$\forall\ u \in D(a_p),\ \mathrm{Re}(a_p(u,u)) = \|\mu^{-1/2}\,\mathrm{curl}_{\mathrm{Re}(p)}\, u\|^2_{L^2(\Omega_T)^3} - \mathrm{Im}(p)^2\|\mu^{-1/2}\,u_T\|^2_{L^2(\Omega_T)^2}\,.$$

It follows that the real part of a_p is $(\ker \gamma^T_\tau \times H^1_0(\Omega_T))$-coercive with respect to $L^2(\Omega_T; \varepsilon\, dx_T)^3$ and a_p is also a closed sectorial sesquilinear form. Next, as we have already remarked for m_p, $p \mapsto a_p(u,u)$ is holomorphic for any fixed $u \in D(a_p)$. Moreover, $D(a_p)$ does not depend on p anymore, so

$$\{a_p, p \in \mathbb{C}\} \text{ is a holomorphic family of type (a) in Kato's sense.} \quad (2.15)$$

Now, using the Green formula (2.13), we get $(a_p)^* = a_{\bar{p}}$, and it derives from (2.15) that

$$\{A_p, p \in \mathbb{C}\} \text{ is a holomorphic family of self-adjoint operators of type (B).} \quad (2.16)$$

Next, for any real p, the orthogonal decomposition (2.10) together with the Poincaré lemma lead to $\ker A_p = \nabla_p H_0^1(\Omega_T) = \{(\partial_{x_1}\varphi, \partial_{x_2}\varphi, -ip\varphi),\ \varphi \in H_0^1(\Omega_T)\}$. The continuous spectrum of A_p is also $\{0\}$ and the discrete spectrum, $\sigma_d(A_p)$, is equal to $\sigma(M_p)$. Finally, Lemma 2.1 warrants that $\sigma_d(A_p)$ does not vanish in the continuous spectrum. So, according to (2.16), the final result can be derived (see Ref. 15) from Remark 4.22 and Theorem 3.9 of Ref. 8 about holomorphic family of self-adjoint operators with compact resolvent. □

2.4. *Asymptotic properties of* $(\lambda_n)_{n\geq 1}$

For any $n \geq 1$ and $p \in \mathbb{R}$, we have $\lambda_n(p) \geq \Lambda_1(p)$, thus, it arises from (2.8) that

$$\forall\ n \geq 1,\ \lim_{|p|\to\infty} \lambda_n(p) = +\infty\,. \tag{2.17}$$

Moreover, by writing $a_p(\varphi_n(p))$ instead of $a_p(\varphi_n(p), \varphi_n(p))$, the property (2.16) together with the problem VII.4.19 of Ref. 8 warrant that $\lambda_n'(p) = a_p'(\varphi_n(p))$. Also, it follows from (2.14) that

$$\begin{aligned}\lambda_n'(p) = 2[p&\|\mu^{-1/2}(\varphi_n)_T(p)\|^2_{L^2(\Omega_T)^2}\\ &- \operatorname{Im}((\mu^{-1}(\varphi_n)_T(p), \nabla_T(\varphi_n)_3(p))_{L^2(\Omega_T)^2})]\,.\end{aligned} \tag{2.18}$$

Now, the expression of $a_p(\varphi_n(p))$ simply gives

$$\forall\ p \in \mathbb{R}^*,\ \left(\frac{\lambda_n(p)}{p}\right)' = \|\mu^{-1/2}(\varphi_n)_T(p)\|_{L^2(\Omega_T)^2} - p^{-2}g(p)\,,$$

with $g(p) = \|\mu^{-1/2}\operatorname{curl}_T(\varphi_n)_T(p)\|^2_{L^2(\Omega_T)} + \|\mu^{-1/2}\nabla_T(\varphi_n)_3(p)\|^2_{L^2(\Omega_T)^2}$. Next, the Poincaré lemma warrants that $g(p) > 0$ for any $p \neq 0$, so we finally have:

$$\forall p \in \mathbb{R}^*,\ \left(\frac{\lambda_n(p)}{p}\right)' < c^2_{\max}\,. \tag{2.19}$$

As we will now see with Proposition 2.2, this inequality is very useful to describe the behavior of the family $(\lambda_n)_{n\geq 1}$ on any compact subset of $\mathbb{R}$.

Proposition 2.2. *The dispersion curves are uniformly converging to infinity as n goes to infinity*:

$$\lim_{n\to\infty}\left(\inf_{p\in\mathbb{R}} \lambda_n(p)\right) = +\infty\,.$$

Proof. Let $n \geq 1$. For any $p \in \mathbb{R}$, we deduce from (2.18) that

$$|\lambda_n'(p)| \leq (2|p|+1)\|\mu^{-1/2}(\varphi_n)_T(p)\|^2_{L^2(\Omega_T)^2} + \|\mu^{-1/2}\,\nabla_T(\varphi_n)_3(p)\|^2_{L^2(\Omega_T)^2}\,. \tag{2.20}$$

But $a_p(\varphi_n(p)) = (A_p\varphi_n(p), \varphi_n(p))_{L^2(\Omega_T)^3} = \lambda_n(p)$, and it can be derived from (2.14) that

$$\begin{aligned}\|\mu^{-1/2}&(\nabla_T(\varphi_n)_3(p) + ip(\varphi_n)_T(p))\|^2_{L^2(\Omega_T)^2}\\ &= \|\mu^{-1/2}\operatorname{curl}_p \varphi_n(p)\|^2_{L^2(\Omega_T)^3} - \|\operatorname{curl}_T(\varphi_n)_T(p)\|^2_{L^2(\Omega_T)^2}\,.\end{aligned}$$

Thus, as $a_p(\varphi_n(p)) = \|\mu^{-1/2}\operatorname{curl}_p \varphi_n(p)\|^2_{L^2(\Omega_T)^3}$, we obtain

$$\|\mu^{-1/2}(\nabla_T(\varphi_n)_3(p) + ip(\varphi_n)_T(p))\|^2_{L^2(\Omega_T)^2} = \lambda_n(p) - \|\operatorname{curl}_T(\varphi_n)_T(p)\|^2_{L^2(\Omega_T)^2}\,, \tag{2.21}$$

for any $p \in \mathbb{R}$.

Next, $\|\mu^{-1/2}(\nabla_T(\varphi_n)_3(p) + ip(\varphi_n)_T(p))\|_{L^2(\Omega_T)^2} + |p|\|\mu^{-1/2}(\varphi_n)_T(p)\|_{L^2(\Omega_T)^2}$ is greater than $\|\mu^{-1/2}\nabla_T(\varphi_n)_3(p)\|_{L^2(\Omega_T)^2}$, and it also follows from (2.20) and (2.21) that:

$$\forall\, t \in \mathbb{R},\ |\lambda_n'(t)| \leq c^2_{\max}(1 + 2|t| + 2t^2) + 2\lambda_n(t)\,. \tag{2.22}$$

Assume now that $p > 0$, the case $p < 0$ being treated in the same way. For any $q \in [0,p]$, we have $|\lambda_n(q) - \lambda_n(0)| \leq \int_0^q |\lambda_n(t)'|\,dt$, so the integration of (2.22) along $[0,q]$ gives

$$|\lambda_n(q) - \lambda_n(0)| \leq c^2_{\max}\left(1 + q + \frac{2}{3}q^2\right) + 2\lambda_n(0)q + 2\int_0^q |\lambda_n(t) - \lambda_n(0)|\,dt\,. \tag{2.23}$$

By setting $f(q) = e^{-2q}\int_0^q |\lambda_n(t) - \lambda_n(0)|\,dt$, this inequality becomes:

$$\forall\, q \in [0,p],\ f'(q) \leq c^2_{\max} q e^{-2q}\left(1 + q + \frac{2}{3}q^2\right) + 2\lambda_n(0)qe^{-2q}\,. \tag{2.24}$$

Now, by integrating (2.24) along $[0,p]$, we get

$$\int_0^p |\lambda_n(q) - \lambda_n(0)|\,dq \leq e^{2p}\left(c^2_{\max}\int_0^p qe^{-2q}\left(1 + q + \frac{2}{3}q^2\right)dq + 2\lambda_n(0)\int_0^p qe^{-2q}dq\right),$$

and, if we replace $\int_0^p |\lambda_n(q) - \lambda_n(0)|\,dq$ by its upper bound in (2.23), we finally obtain

$$\begin{aligned} &|\lambda_n(p) - \lambda_n(0)| \\ &\quad\leq c^2_{\max}\left\{|p| + p^2 + \frac{2}{3}|p|^3 + 2e^{2|p|}\left|\int_0^p e^{-2|t|}\left(|t| + t^2 + \frac{2}{3}|t|^3\right)dt\right|\right\} \\ &\qquad + 2\lambda_n(0)\left(|p| + 2e^{2|p|}\int_0^p te^{-2|t|}\,dt\right)\,. \end{aligned} \tag{2.25}$$

To prove Proposition 2.2 now, assume for the moment that a real sequence $(p_n)_{n\geq 1}$ satisfying

$$\forall\, n \geq 1,\ \lambda_n(p_n) \leq M\,, \tag{2.26}$$

can be found. In this case, $(p_n)_{n\geq 1}$ converges to 0. Indeed, we could otherwise find $\eta > 0$ and a subsequence $(p_{n_k})_{k\geq 1}$ of $(p_n)_{n\geq 1}$, such that $|p_{n_k}| > \eta$ for any $k \geq 1$. Then, by setting $P = c^{-1}_{\min} M^{1/2}$, it would also follow from (2.8) that

$$\forall\, k \geq 1\,,\quad \eta < |p_{n_k}| \leq P\,. \tag{2.27}$$

Instead of considering a subsequence of $(p_{n_k})_{k\geq 1}$, we may assume that the p_{n_k}, $k \geq 1$, are all positive or negative numbers. If $p_{n_1} > 0$, for example, we have $0 < p_{n_k} < P$ for any $k \geq 1$, and (2.19) implies

$$\frac{\lambda_{n_k}(P) - c_{\max}^2 P^2}{P} < \frac{\lambda_{n_k}(p_{n_k}) - c_{\max}^2 p_{n_k}^2}{p_{n_k}} .$$

Thus, we derive immediately from (2.26) and (2.27):

$$\forall\, k \geq 1 , \quad \frac{\lambda_{n_k}(P) - c_{\max}^2 P^2}{P} < M\eta^{-1} + c_{\max}^2 P . \tag{2.28}$$

But, this inequality is not possible because $\lim_{k\to\infty} \lambda_{n_k}(P) = +\infty$. The same method applies to a negative p_{n_1}. We have also proved that

$$\lim_{n\to\infty} p_n = 0 . \tag{2.29}$$

Thus, for any $n \in \mathbb{N}^*$, we derive from (2.25) that

$$\begin{aligned} &\left| \frac{\lambda_n(p_n)}{\lambda_n(0)} - 1 \right| \\ &\quad \leq \frac{c_{\max}^2}{\lambda_n(0)} \left\{ |p_n| + p_n^2 + \frac{2}{3}|p_n|^3 + 2e^{2|p_n|} \left| \int_0^{p_n} e^{-2|t|} \left(|t| + t^2 + \frac{2}{3}|t|^3 \right) dt \right| \right\} \\ &\quad + 2 \left(|p_n| + 2e^{2|p_n|} \int_0^{p_n} t e^{-2|t|}\, dt \right) . \end{aligned} \tag{2.30}$$

Next, as $\lim_{n\to\infty} \lambda_n(0) = +\infty$, the inequality (2.26) involves $\lim_{n\to\infty} |\frac{\lambda_n(p_n)}{\lambda_n(0)} - 1| = 1$. But (2.29) proves that the second term of (2.30) tends to 0 as n goes to infinity. This contradiction shows that assumption (2.26) is wrong. Therefore, Proposition 2.2 follows. ☐

2.5. *Spectral representation and thresholds of M*

For any $(n, p) \in \mathbb{N}^* \times \mathbb{R}$, the function $\phi_n(p) : (x_T, x_3) \mapsto (2\pi)^{-1/2} \varphi_n(x_T) e^{ipx_3}$ defined on Ω, is a generalized eigenfunction of M. Indeed, $\phi_n(p)$ belongs to $D(M)_{\text{loc}} = \{u,\ (x_T, x_3) \mapsto \psi(x_3) u(x_T, x_3) \in D(M),\ \forall\, \psi \in C_0^\infty(\mathbb{R})\}$, and it satisfies in addition

$$(\mathcal{M}\phi_n(p))(x) = \lambda_n(p)(\phi_n(p))(x) \quad \text{a.e. } x \in \Omega.$$

Moreover, for every f belonging to H_ε,

$$X \mapsto \int_{\{(x_T,x_3)\in\Omega,\ |x_3|<X\}} \varepsilon(x_T)(f(x_T, x_3), (\phi_n(p))(x_T, x_3))_{\mathbb{C}^3}\, dx_T\, dx_3$$

has a limit $\tilde{f}_n(p)$ in $L^2(\mathbb{R})$ as X tends to infinity, and it can be derived from (2.3) that $\mathcal{U} : f \mapsto (\tilde{f}_n)_{n\geq 1}$ is a unitary transform from H_ε onto $\oplus_{n\geq 1} L^2(\mathbb{R})$ (see Ref. 15). The set $\{\phi_n(p), (n, p) \in \mathbb{N}^* \times \mathbb{R}\}$ is also a complete family in H_ε. At last, $\mathcal{U}$ reduces the operator M, because $\mathcal{U}M\mathcal{U}^*$ is the multiplication operator L in $Y = \oplus_{n\geq 1}L^2(\mathbb{R})$ by the family $(\lambda_n(p))_{p\geq 1}$:

$$\begin{cases} D(L) = \{(f_n)_{n\geq 1} \in Y,\ (\lambda_n f_n)_{n\geq 1} \in Y\} \\ \forall\ f = (f_n)_{n\geq 1} \in Y\,, \quad Lf = (\lambda_n f_n)_{n\geq 1}\,. \end{cases}$$

In particular, we deduce from Ref. 2:

Proposition 2.3. *The spectrum of M is absolutely continuous and*

$$\sigma(M) = \overline{\{\lambda_n(p),\ p \in \mathbb{R},\ n \geq 1\}}\,.$$

Furthermore, the spectral theory yields as in Ref. 5 that $R_M(z) = \mathcal{U}^* R_L(z)\mathcal{U}$ for any $z \in \mathbb{C}^{\pm}$. Thus, we have

$$\forall\ z \in \mathbb{C}^{\pm},\ \forall\ (f,g) \in H^2_\varepsilon,\ (R_M(z)f,g)_{H_\varepsilon} = \sum_{n\geq 1} r_n(z)\,, \tag{2.31}$$

where $r_n(z) = \int_{\mathbb{R}} \frac{h_n(p)}{\lambda_n(p)-z}dp$ with $h_n(p) = \tilde{f}_n(p)\overline{\tilde{g}_n(p)}$, $n \geq 1$, $p \in \mathbb{R}$.

Recalling (2.18), we remark that the sign of $\lambda_n'(p)$, $n \geq 1$, is not easily predictable for any $p \in \mathbb{R}$. Indeed, contrary to the acoustic operator (see Ref. 1), it is not yet known if the dispersion curves of M are monotone on $\mathbb{R}^*_{\pm}$. But, Proposition 2.1 and (2.17) warrant that every λ_n', $n \geq 1$, is a non-uniformly vanishing analytic function on $\mathbb{R}$. Hence, the set $\mathcal{P}_n = \{p \in \mathbb{R},\ \lambda_n'(p) = 0\}$ is discrete for any $n \geq 1$ and it is also at most countable. Call N_n^+ the cardinal of $\mathcal{P}_n \cap \mathbb{R}_+$ and set $J_n^+ = \{j \in \mathbb{N},\ j < N_n^+\}$. There is a strictly increasing function $j \mapsto p_j^n$ such that $\mathcal{P}_n \cap \mathbb{R}_+ = \{p_j^n, j \in J_n^+\}$. When N_n^+ is finite, we set $p_{N_n^+} = +\infty$ and $\bar{J}_n^+ = J_n^+ \cup \{N_n^+\}$. Otherwise, $\bar{J}_n^+$ denotes simply J_n^+. In the same way, we define N_n^- as the cardinal of $\mathcal{P}_n \cap \mathbb{R}^*_-$ and J_n^- as the set $\{j \in \mathbb{Z}^*_-,\ j \geq -N_n^-\}$. Again, there is a strictly increasing function $j \mapsto p_j^n$ such that $\mathcal{P}_n \cap \mathbb{R}^*_- = \{p_j^n, j \in J_n^-\}$. Next, we set $p_{-N_n^- -1} = -\infty$ and $\bar{J}_n^- = \{-N_n^- - 1\} \cup J_n^-$ when N_n^- is finite, and $\bar{J}_n^- = J_n^-$ otherwise. Recalling all these notations, the set $\mathcal{P}_n$ is equal to $\{p_j^n,\ j \in J_n\}$ with $J_n = J_n^- \cup J_n^+$, and we have $p_j^n < p_{j+1}^n$ for any $(j, j+1) \in J_n^2$.

Each real number $\lambda_j^n = \lambda_n(p_j^n)$, $j \in J_n$, $n \geq 1$, is a "threshold" of M, and the set of all the thresholds, $\{\lambda_j^n,\ n \geq 1,\ j \in J_n\}$, is denoted by $\mathcal{T}$. In fact, this approach generalizes the usual definition of the thresholds of the acoustic operator used in Ref. 3. Every p_j^n, $n \geq 1$ and $j \in J_n$, is also a zero of $p \mapsto \lambda_n(p) - \lambda_j^n$ with multiplicity $N_j^n \geq 2$. Furthermore, for any $n \geq 1$ and $j \in \overline{J_n} = \{-N_n^- - 1\} \cup J_n$, λ_n is an analytic diffeomorphism from $]p_j^n, p_{j+1}^n[$ onto $I_j^n = \lambda_n(]p_j^n, p_{j+1}^n[)$. In the following, its inverse function is denoted by ξ_j^n.

3. Limiting Absorption Principle for the Maxwell Operator

3.1. *Description of the method*

For any real s, the space H^s_ε defined by

$$H^s_\varepsilon = H_\varepsilon \cap \{u \text{ is measurable},\ (x_T, x_3) \mapsto (1 + x_3^2)^{s/2} u(x_T, x_3) \in L^2(\Omega)^3\}$$

is Hilbertian for the scalar product of $L^2(\Omega;\ \varepsilon(1+x_3^2)^s\,dx_T\,dx_3)^3$. When $s>0$, the imbeddings $H_\varepsilon^{-s}\hookrightarrow H_\varepsilon\hookrightarrow H_\varepsilon^s$ are continuous, so $z\mapsto R_M^{\pm}(z)=R_M(z)$ can be seen as an analytic function from $\mathbb{C}^{\pm}$ into $B(H_\varepsilon^s,H_\varepsilon^{-s})$. For any compact subset K of $\mathbb{C}$, we will prove in the following that this resolvent function $R_M^{\pm}$ can be extended continuously to $K\cap\overline{\mathbb{C}^{\pm}}$.

The method consists of finding two real positive numbers δ and C_M, both independent of the functions f and g in H_ε^s for some convenient $s>0$, which verify

$$|(R_M^{\pm}(z')f,g)_{H_\varepsilon}-(R_M^{\pm}(z)f,g)_{H_\varepsilon}|\le C_M\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}|z-z'|^{\delta}\tag{3.32}$$

for any z and z' in $K\cap\overline{\mathbb{C}^{\pm}}$. Next, as $H_\varepsilon^{-s}\subset(H_\varepsilon^s)'$ and $\langle u,v\rangle_{(H_\varepsilon^s)',H_\varepsilon^s}=(u,v)_{H_\varepsilon}$ for any $u\in H_\varepsilon^{-s}$ and $v\in H_\varepsilon^s$, it follows readily from (3.32):

$$\forall\ (z,z')\in(K\cap\mathbb{C}^{\pm})^2,\ \|R_M^{\pm}(z')f-R_M^{\pm}(z)f\|_{H_\varepsilon^{-s}}\le C_M\|f\|_{H_\varepsilon^s}|z-z'|^{\delta}\,.$$

At last, K being a closed set, the general extension theorem of uniformly continuous functions finally involves

$$\forall\ (z,z')\in(K\cap\overline{\mathbb{C}^{\pm}})^2,\ \|R_M^{\pm}(z')-R_M^{\pm}(z)\|_{B(H_\varepsilon^s,H_\varepsilon^{-s})}\le c|z'-z|^{\delta}\,,\tag{3.33}$$

and $R_M^{\pm}$ is also continuous on $K\cap\overline{\mathbb{C}^{\pm}}$. Let us see now that (3.32) can actually be satisfied by convenient functions f and g in H_ε^s.

3.2. *Singular Cauchy integrals*

Recalling (2.31), any function r_n, $n\ge1$, also has to be extended continuously to $K\cap\overline{\mathbb{C}^{\pm}}$, K being a compact subset of $\mathbb{C}$. But, for any $n\in\mathbb{N}_K=\{n\ge 1,\ \lambda_n^{-1}(K\cap\mathbb{R})\neq\emptyset\}$, the corresponding Cauchy integral $z\mapsto\int_{\mathbb{R}}\frac{h_n(p)}{\lambda_n(p)-z}dp$ may be "singular" on $K\cap\mathbb{R}$.

However, this is not the case when n belongs to $\mathbb{N}^*\backslash\mathbb{N}_K$. Indeed, as $(p,z)\mapsto|\lambda_n(p)-z|$ is continuous on the compact set $(K\cap\mathbb{R})\times K$, we have $\inf\{|\lambda_n(p)-z|,\ p\in\mathbb{R},z\in K\}>0$. Moreover, according to (2.17), $\lambda_n^{-1}(K\cap\mathbb{R})$ is bounded, thus we deduce from Proposition 2.2:

$$d=\inf_{n\in\mathbb{N}^*\backslash\mathbb{N}_K}\{|\lambda_n(p)-z|,\ p\in\mathbb{R},z\in K\}>0.\tag{3.34}$$

Next, taking f and g in H_ε^s for some $s>1/2$, it is proved in Ref. 3 that

$$\forall\ n\ge1,\ \forall\ p\in\mathbb{R},\ |h_n(p)|\le A\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}\,,\tag{3.35}$$

where the constant A depends neither on f nor g. Hence, the Parseval equality combined with (3.34) and (3.35) proves that $\sum_{n\in\mathbb{N}^*\backslash\mathbb{N}_K}r_n$ is a Lipschitz continuous function on $K\cap\mathbb{C}^{\pm}$, because we have

$$\begin{gathered}\forall\ (z,z')\in(K\cap\mathbb{C}^{\pm})^2\,,\\ \left|\sum_{n\in\mathbb{N}^*\backslash\mathbb{N}_K}r_n(z')-\sum_{n\in\mathbb{N}^*\backslash\mathbb{N}_K}r_n(z)\right|\le\frac{A\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}}{d^2}|z'-z|\,.\end{gathered}\tag{3.36}$$

Let study now the case $n \in \mathbb{N}_K$. For any $z \in K \cap \mathbb{C}^{\pm}$, the complex number $r_n(z)$ decomposes simply in $\sum_{j \in \overline{J_n}} r_n^j(z)$, with $r_n^j(z) = \int_{p_j^n}^{p_{j+1}^n} \frac{h_n(p)}{\lambda_n(p)-z} dp$. This leads to introduce the set $J_n^K = \{j \in \overline{J_n},\ K \cap \overline{I_j^n} \neq \emptyset\}$, because we know that $\inf\{|\lambda_n(p) - z|,\ p \in \overline{]p_j^n, p_{j+1}^n[},\ z \in K\} > 0$ for any j in $\overline{J_n} \backslash J_n^K$. Indeed, this inequality, which is obvious when $[p_j^n, p_{j+1}^n]$ is bounded, follows directly from (2.17) when $p_j^n = -\infty$ or $p_{j+1}^n = +\infty$. Furthermore, for any fixed $X > 0$, the set $\{j \in \overline{J_n},\ |p_j^n| \leq X\}$ is at most a finite, and $\lambda_n^{-1}(K \cap \mathbb{R})$ is bounded, so we get

$$d_n = \inf\{|\lambda_n(p) - z|,\ j \in \overline{J_n} \backslash J_n^K,\ p \in \overline{]p_j^n, p_{j+1}^n[},\ z \in K\} > 0\,.$$

Hence, (3.35) and the Parseval equality involve in the same way as with (3.36):

$$\forall\ (z, z') \in (K \cap \mathbb{C}^{\pm})^2\,,$$

$$\left| \sum_{j \in \overline{J_n} \backslash J_n^K} r_n^j(z') - \sum_{j \in \overline{J_n} \backslash J_n^K} r_n^j(z) \right| \leq \frac{A\|f\|_{H_\varepsilon^s} \|g\|_{H_\varepsilon^s}}{d_n^2} |z' - z|\,. \tag{3.37}$$

Next, taking account of (2.17) and Proposition 2.2, we remark that $\mathbb{N}_K$ and J_n^K, for n in $\mathbb{N}_K$, are both at most finite sets. Therefore, it is sufficient to examine $z \mapsto r_n^j(z)$ for some fixed n in $\mathbb{N}_K$ and $j \in J_n^K$. In this case, by setting $\lambda = \lambda_n(p)$ in the expression of $r_n^j(z)$, for $z \in K \cap \mathbb{C}$, we obtain:

$$r_n^j(z) = \int_{I_j^n} \frac{H_j^n(\lambda)}{\lambda - z} d\lambda \quad \text{with} \quad H_j^n(\lambda) = \frac{(h_n \circ \xi_j^n)(\lambda)}{(\lambda_n' \circ \xi_j^n)(\lambda)}\,. \tag{3.38}$$

In view of extending $z \mapsto r_n^j(z)$ to $K \cap \mathbb{R}$, we will refer to the following corollary of the Korn–Privaloff theorem (see Ref. 6 or 11):

Lemma 3.1. *Let $\delta \in [0, 1[$, $(a, b) \in \mathbb{R}^2$ with $a < b$, and h be a δ-Hölder continuous function on the compact set $[a, b]$:*

$$\exists\ A_h \geq 0,\ \forall\ (\lambda, \lambda') \in [a, b]^2,\ |h(\lambda') - h(\lambda)| \leq A_h |\lambda' - \lambda|^{\delta}\,.$$

Assume that $h(a) = h(b) = 0$ and set $\mathcal{H}h^{\pm}(z) = \int_a^b \frac{h(\lambda)}{\lambda - z} d\lambda$ for any $z \in \mathbb{C}^{\pm}$. Then, for every $\tau \in [a, b]$, the following limits exist

$$\lim_{z \to \tau, \pm \mathrm{Im}(z) > 0} \mathcal{H}h^{\pm}(z) = p.v. \left(\int_a^b \frac{h(\lambda)}{\lambda - \tau} d\lambda \right) \pm i\pi h(\tau)\,,$$

and $\mathcal{H}h^{\pm}$ is locally δ-Hölder continuous on $V^{\pm} = \{z \in \overline{\mathbb{C}^{\pm}},\ a \leq \mathrm{Re}(z) \leq b\}$. Indeed, there is a continuous function c on $(V^{\pm})^2$, that does not dependent on h, such that

$$\forall\ (z, z') \in (V^{\pm})^2,\ |\mathcal{H}h^{\pm}(z') - \mathcal{H}h^{\pm}(z)| \leq A_h c(z, z') |z' - z|^{\delta}\,.$$

Recalling (3.38), $z \mapsto r_n^j(z)$ can be extended continuously to the real axis with Lemma 3.1, when H_j^n is a Hölder continuous function. This condition can be filled by choosing f and g in convenient H_ε^s-spaces. Indeed, when $s > 1/2$, it is proved in

Ref. 3 that h_n is locally Hölder continuous: For any $\delta \in [0,1]$ such that $\delta < s - 1/2$, there is a continuous function A_n on $\mathbb{R}^2$, independent of f or g, such that,

$$\forall\ (p,p') \in \mathbb{R}^2,\ |h_n(p') - h_n(p)| \leq A_n(p,p')\|f\|_{H^s_\varepsilon}\|g\|_{H^s_\varepsilon}|p'-p|^\delta\,. \tag{3.39}$$

But, as $\lambda'_n \circ \xi^n_j$ vanishes on $K \cap \mathcal{T}$, we will first examine the case of a compact set K of $\mathbb{C}\backslash\mathcal{T}$ (limiting absorption principle outside the thresholds). Next, a limiting absorption principle for M in a neighborhood of the thresholds (limiting absorption principle at the thresholds) will be deduced.

3.3. *Limiting absorption principle outside the thresholds*

As mentioned earlier, we assume now that K is a compact set of $\mathbb{C}\backslash\mathcal{T}$, the functions f and g being in H^s_ε for some $s > 1/2$. In the following, δ denotes a real number in $[0,1]$ such that $\delta < s - 1/2$, and we recall that (n,j) belongs to $\mathbb{N}_K \times J^K_n$.

Now, as K is a compact set of $\mathbb{C}\backslash\mathcal{T}$, we can find some $\eta > 0$ such that $\lambda^{-1}(K \cap \mathbb{R}) \cap]p^n_j, p^n_{j+1}[$ is included in $]p^n_j + 3\eta, p^n_{j+1} - 3\eta[$. Next, consider $\beta_0 \in C^\infty_0(]p^n_j + \eta, p^n_{j+1} - \eta[)$ such that $\beta_0(p) = 1$ for any $p \in [p^n_j + 2\eta, p^n_{j+1} - 2\eta]$, and set $\beta_1(p) = 1 - \beta_0(p)$ for every $p \in \mathbb{R}$, β_0 being 0 outside $]p^n_j + \eta, p^n_{j+1} - \eta[$. Then, for every $z \in K \cap \mathbb{C}^\pm$, $r^j_n(z)$ decomposes in $r^{j,0}_n(z) + r^{j,1}_n(z)$ with $r^{j,k}_n(z) = \int_{p^n_j}^{p^n_{j+1}} \frac{\beta_k(p)h_n(p)}{\lambda_n(p)-z}dp$, $k \in \{0,1\}$.

Also, by setting $\lambda = \lambda_n(p)$ in $r^n_{j,0}(z)$, we get $r^{j,0}_n(z) = \int_{I^n_j} \frac{\psi^n_j(\lambda)}{\lambda - z}d\lambda$, where $\psi^n_j = H^n_j \times (\beta_0 \circ \xi^n_j)$. In fact, in the previous expression of $r^{j,0}_n$, I^n_j can be considered as a bounded set. Indeed, for any $z \in K \cap \mathbb{C}^\pm$, the complex number $r^{j,0}_n(z)$ is not modified by eventually replacing p^n_j by $\inf(\mathrm{supp}(\beta_0)) - \eta$ if $p^n_j = -\infty$, and p^n_{j+1} by $\sup(\mathrm{supp}(\beta_0)) + \eta$ if $p^n_{j+1} = +\infty$. Moreover, $\mathrm{supp}(\beta_0)$ is a compact subset of $]p^n_j, p^n_{j+1}[$ and the derivative λ'_n does not vanish on $\mathrm{supp}(\beta_0)$, hence $p \mapsto \varphi_0(p)|\lambda'_n(p)|^{-1}$ is a Lipschitz continuous function on $\mathrm{supp}(\beta_0)$. Next, $\bar{I} = \lambda_n(\mathrm{supp}(\beta_0))$ being a compact subset of I^n_j, ξ^n_j is Lipschitz continuous on $\bar{I}$. Finally, it follows from (3.39) that ψ^n_j is a δ-Hölder continuous function on $\bar{I}$. Finally ψ^n_j vanishes at $\overline{I^n_j}\backslash I^n_j$, so, according to Lemma 3.1, a constant c^j_n, independent of f and g, can be found, such that

$$\forall\ (z,z') \in (K \cap \mathbb{C}^\pm)^2,\ |r^{j,0}_n(z') - r^{j,0}_n(z)| \leq c^j_n\|f\|_{H^s_\varepsilon}\|g\|_{H^s_\varepsilon}|z-z'|^\delta\,. \tag{3.40}$$

Next, λ_n is a strictly monotone function on $]p^n_j, p^n_{j+1}[$ and $\inf\{|p-q|,\ p \in \mathrm{supp}(\beta_1),\ q \in \lambda^{-1}_n(K \cap \mathbb{R}) \cap \overline{]p^n_j, p^n_{j+1}[}\} \geq \eta$, so we have

$$\inf\{|\lambda_n(p) - z|,\ p \in \mathrm{supp}(\beta_1),\ z \in K \cap\ \overline{I^n_j}\} > 0\,.$$

Therefore, $r^{j,1}_n$ is Lipschitz continuous on $K \cap \mathbb{C}^\pm$. Moreover, $(z,z') \mapsto |z'-z|^{1-\delta}$ is continuous on the compact set K^2, so, according to (3.40), we can find a constant C^j_n depending neither on f nor g, which satisfies

$$\forall\ (z,z') \in (K \cap \mathbb{C}^\pm)^2,\ |r^j_n(z') - r^j_n(z)| \leq C^j_n\|f\|_{H^s_\varepsilon}\|g\|_{H^s_\varepsilon}|z-z'|^\delta\,. \tag{3.41}$$

Thus, for any z and z' in $K\cap\mathbb{C}^{\pm}$, we deduce from (3.36), (3.37) and (3.41), that

$$|(R_M^{\pm}(z')f,g)_{H_\varepsilon^s} - (R_M^{\pm}(z)f,g)_{H_\varepsilon^s}| \le C_M\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}|z-z'|^{\delta}\,, \tag{3.42}$$

where C_M is independent of f and g. As seen in Sec. 3.1, we finally have:

Theorem 3.1. *Let K be a compact subset of $\mathbb{C}\backslash\mathcal{T}$. Then, $z\mapsto R_M^{\pm}(z)$ extends continuously to $K\cap\overline{\mathbb{C}^{\pm}}$ in a Hölder continuous function. Indeed, for any $\delta\in[0,1]$ satisfying $\delta<s-1/2$, a constant $C_M=C_M(s,\delta,K)$ can be found, such that*

$$\|R_M^{\pm}(z')-R_M^{\pm}(z)\|_{B(H_\varepsilon^s,H_\varepsilon^{-s})}\le C_M|z'-z|^{\delta},\ \forall\ z,z'\in K\cap\ \overline{\mathbb{C}^{\pm}}.$$

Moreover, for every $\tau\in\mathbb{R}\backslash\mathcal{T}$, $\{\tau\}$ is a compact subset of $\mathbb{C}\backslash\mathcal{T}$. Thus, under the assumptions of Theorem 3.1 and Lemma 3.1, we have:

$$\begin{aligned}&\lim_{z\to\tau,\pm\mathrm{Im}(z)>0}(R_M^{\pm}(z)f,g)_{H_\varepsilon}\\ &\qquad=\sum_{n\ge1}\left\{p.v.\left(\int_{\mathbb{R}}\frac{h_n(p)}{\lambda_n(p)-\tau}dp\right)\pm i\pi\sum_{q\in\lambda_n^{-1}(\{\tau\})}\frac{h_n(q)}{|\lambda_n'(q)|}\right\}.\end{aligned}$$

3.4. *Limiting absorption principle at the thresholds*

Let $\tau\in\mathcal{T}$ and K be a compact subset of $(\mathbb{C}\backslash\mathcal{T})\cup\{\tau\}$. As we already remarked in Sec. 3.1, it remains to find convenient functions f and g such that any r_n^j, $n\in\mathbb{N}_K$ and $j\in J_n^K$, is a Hölder continuous function on $K\cap\mathbb{C}^{\pm}$. But, recalling (3.38), the denominator of H_j^n vanishes on K (at $\lambda=\lambda_j^n$) when j belongs to $J_n'(\tau)=\{m\in J_n^K,\ \lambda_m^n=\tau\}$. To make the analysis of this problem easier, we first start by decomposing $\sum_{j\in J_n^K}r_n^j$.

3.4.1. *An adapted partition of J_n^K*

Note that the set $J_n'(\tau)$ may be empty, but $\cup_{n\in\mathbb{N}_K}J_n'(\tau)\neq\emptyset$ because τ belongs to $\mathcal{T}$. Furthermore, $p_n^j\in\overline{]p_{j\pm1}^n,p_j^n[}$, so $j\pm1$ is not in $J_n'(\tau)$ when j belongs to $J_n'(\tau)$. Recalling the strict monotonity of λ_n on $]p_{j\pm1}^n,p_j^n[$, we also have $\lambda_n(p_{j\pm1}^n)\neq\tau$. Hence, $j\pm1$ does not belong to $J_n'(\tau)$, and the sets $J_n'(K)=\{j\in J_n^K,\ \tau\notin\overline{I_j^n}\backslash I_j^n\}$ and $\cup_{n\in J_n'(\tau)}\{j-1,j\}$ also define a partition of J_n^K. Thus, for any $z\in K\cap\mathbb{C}^{\pm}$, we can write:

$$\sum_{j\in J_n^K}r_n^j(z)=\sum_{j\in J_n'(\tau)}\left(r_n^{j-1}(z)+r_n^j(z)\right)+\sum_{j\in J_n'(K)}r_n^j(z)\,. \tag{3.43}$$

Now, for any $j\in J_n'(K)$, there is some $\eta>0$ such that

$$\lambda_n^{-1}(K\cap\mathbb{R})\cap\overline{]p_j^n,p_{j+1}^n[}\subset\,]p_j^n+3\eta,p_{j+1}^n-3\eta[\,. \tag{3.44}$$

Indeed, we could otherwise build a sequence $(p_k)_{k\ge1}$ in $\lambda^{-1}(K\cap\mathbb{R})\cap\overline{]p_j^n,p_{j+1}^n[}$, such that $\min(p_k-p_j^n,p_{j+1}^n-p_k)\le1/k$ for any $k\ge1$. As $\lambda^{-1}(K\cap\mathbb{R})$ is a compact set, $(p_k)_{k\geq 1}$ can be assumed to converge to p_m^n for some m in $\{j, j+1\}$. Thus, $(\lambda_n(p_k))_{k\geq 1}$ should converge to λ_m^n with $m \in J_n$. Finally, K being closed, the real λ_m^n should belong to $K \cap \mathcal{T}$. But this is impossible because $m \notin J_n'(\tau)$. Hence, (3.44) is valid.

According to (3.44), the method used in Sec. 3.2 for r_m^k, with $m \in \mathbb{N}_K$ and $k \in J_m^K$, applies here without essential modification to every r_n^j, $j \in J_n'(K)$. Hence, for any $s > 1/2$, we can find a constant c_n^j depending neither on f nor g, which satisfies

$$\forall\ (z, z') \in (K \cap \mathbb{C}^{\pm})^2,\ |r_n^j(z') - r_n^j(z)| \leq c_n^j \|f\|_{H_\varepsilon^s} \|g\|_{H_\varepsilon^s} |z - z'|^{\delta}\,. \tag{3.45}$$

Now, let us study $z \mapsto R_n^j(z) = r_n^{j-1}(z) + r_n^j(z)$ for every $j \in J_n'(\tau)$. In this case, we have $\lambda_j^n = \tau$ and $R_n^j(z) = \int_{p_{j-1}^n}^{p_{j+1}^n} \frac{h_n(p)}{\lambda_n(p)-z} dp$ for any $z \in K \cap \mathbb{C}^{\pm}$. The function $p \mapsto \frac{h_n(p)}{\lambda_n(p)-z}$ also has to be extended at $p = p_j^n$. As $\lambda_n'(p_j^n) = 0$, this extension necessarily requires a vanishing h_n at p_j^n. This condition will be filled by choosing f and g in suitable subspaces of H_ε^s, which are introduced now.

3.4.2. *The NH_ε^s-spaces*

Let $k \in \mathbb{N}$ and $s > k - 1/2$. Then, for any $\varphi \in H_\varepsilon^s$, it is shown in Ref. 15 that each $\tilde{\varphi}^n$, $n \geq 1$, belongs to $C^{k-1}(\mathbb{R})$. Moreover, for all $q \in \mathbb{R}$ and any $\alpha \in \{0, 1, \ldots, k-1\}$, $\varphi \mapsto \frac{d^\alpha \tilde{\varphi}^n}{dp^\alpha}(q)$ is a linear form on H_ε^s. Hence, for any $j \in J_n$,

$$NH_\varepsilon^s(n, j, k) = \left\{ \varphi \in H_\varepsilon^s,\ \frac{d^\alpha \tilde{\varphi}_n}{dp^\alpha}(p_j^n) = 0,\ \forall\ \alpha = 0, 1, \ldots, k-1 \right\}$$

is a closed subspace of H_ε^s. Now, let us see by taking f and g in the previous spaces, that R_j^n, $j \in J_n'(\tau)$, becomes a Hölder continuous function in any neighborhood of τ.

3.4.3. *Hölder continuous properties of R_n^j, $n \in \mathbb{N}_K$, $j \in J_n'(\tau)$*

To see this, let α belong to $\{1, 2, \ldots, N_j^n - 1\}$, and set $\alpha' = N_j^n - \alpha$. Next, fix $s > \alpha - 1/2$ and $s' > \alpha' - 1/2$ such that $\alpha + \alpha' > N_j^n$. At last, take $f \in NH_\varepsilon^s(n, j, \alpha)$, $g \in NH_\varepsilon^{s'}(n', j, \alpha')$ and θ in $[0, \frac{t+t'-1}{2}[$, where $t = \min(1, s - \alpha - 1/2)$ and $t' = \min(1, s' - \alpha' - 1/2)$.

As $j \in J_n'(\tau)$, it is clear that $\lambda_{j\pm 1}^n$ belongs to $\mathcal{T} \backslash \{\tau\}$ if $j \pm 1$ is in J_n and that $\lim_{p \mapsto p_{j\pm 1}^n} \lambda_n(p) = +\infty$ otherwise. Therefore, the inclusion $K \backslash \{\tau\} \subset \mathbb{C} \backslash \mathcal{T}$ allows us to build a compact subset $\bar{P}$ of $\overline{]p_{j-1}^n, p_{j+1}^n[}$, such that $p_j^n \in \bar{P}$, and we also have

$$\delta_n^j = \inf\{|\lambda_n(p) - z|,\ p \in \overline{]p_{j-1}^n, p_{j+1}^n[} \backslash \bar{P},\ z \in K\} > 0\,. \tag{3.46}$$

Take now $\beta_0 \in C_0^\infty(]p_{j-1}^n, p_{j+1}^n[)$ such that $\beta_0(p) = 1$ for any $p \in \bar{P}$, and set $\beta_1(p) = 1 - \beta_0(p)$ for any $p \in \mathbb{R}$, β_0 being 0 outside $]p_{j-1}^n, p_{j+1}^n[$. Then, for every $z \in K \cap \mathbb{C}^{\pm}$, $R_n^j(z)$ decomposes in $R_n^j(z) = R_n^{j,0}(z) + R_n^{j,1}(z)$ with $R_n^{j,k}(z) = \int_{p_{j-1}^n}^{p_{j+1}^n} \frac{\beta_k(p) h_n(p)}{\lambda_n(p)-z} dp$, for each $k = 0, 1$.

Now, $\overline{P_0}$ being $\operatorname{supp}(\beta_0)$, and P_0 denoting the interior set of $\overline{P_0}$, we have

$$R_n^{j,0}(z) = \int_{P_0\cap]p_{j-1}^n,p_j^n[} \frac{\beta_0(p)h_n(p)}{\lambda_n(p)-z}dp + \int_{P_0\cap]p_j^n,p_{j+1}^n[} \frac{\beta_0(p)h_n(p)}{\lambda_n(p)-z}dp\,,$$

for any $z \in K\cap\mathbb{C}^{\pm}$. Next, by setting $p = \xi_{j-1}^n(\lambda)$ in the first integral and $p = \xi_j^n(\lambda)$ in the second one, we get

$$R_n^{j,0}(z) = \sum_{k=j-1}^{j} \int_{I_0\cap I_k^n} \frac{\phi_k(\lambda)}{\lambda - z}d\lambda\,,$$

with $I_0 = \lambda_n(P_0)$, $\phi_k = \psi\circ\xi_k^n$ for $k \in \{j-1, j\}$, and $\psi(p) = \frac{\beta_0(p)h_n(p)}{|\lambda_n'(p)|}$ for any $p \in P_0\backslash\{p_j^n\}$. But $\bar{P}$ is a subset of $\overline{P_0}$, so $p_j^n \in \overline{P_0}$. According to Corollary 3.2.1 of Ref. 15, we can also write

$$\forall\, p \in \overline{P_0},\ h_n(p) = (p - p_j^n)^{N_j^n-1}H(p)\,,$$

where H is a vanishing function at $p = p_j^n$, which satisfies in addition

$$\forall\,(p,p') \in \overline{P_0}^2,\ |H(p') - H(p)| \le A\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}|p'-p|^{\theta}\,, \tag{3.47}$$

for some constant A, independent of f and g. Furthermore, p_j^n is a zero of $\lambda_n - \lambda_j^n$ with multiplicity $N_j^n \ge 2$, so there is an analytic function a_n on $\mathbb{R}$, which never vanishes on $\overline{P_0}$, such that

$$\forall\, p \in \overline{P_0},\ \lambda_n'(p) = (p-p_j^n)^{N_j^n-1}a_n(p)\,.$$

Next, as $H(p_j^n) = 0$, the function ψ can be extended continuously to $\overline{P_0}$ by setting

$$\psi(p) = \begin{cases} \dfrac{(-1)^{N_j^n-1}\beta_0(p)H(p)}{|a_n(p)|} & \text{if } p \in \overline{P_0}\cap\,]p_{j-1}^n, p_j^n]\,,\\[2ex] \dfrac{\beta_0(p)H(p)}{|a_n(p)|} & \text{if } p \in \overline{P_0}\cap[p_j^n, p_{j+1}^n[\,. \end{cases}$$

As $\beta_0 H/|a_n| \in C^1(]p_{j-1}^n, p_{j+1}^n[)$ and $\psi(p_j^n) = 0$, (3.47) furnishes a constant B, independent of f and g, such that

$$\forall\,(p,p') \in \overline{P_0}^2,\ |\psi(p') - \psi(p)| \le B\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}|p'-p|^{\theta}\,. \tag{3.48}$$

Moreover, as $\overline{I_0} = \lambda_n(\overline{P_0})$ is a compact subset of $I_{j-1}^n \cup\{\tau\}\cup I_j^n$, the inverse function ξ_k^n is $1/N_j^n$-Hölder continuous on $\overline{I_0}\cap\overline{I_k^n}$ for every $k \in \{j-1,j\}$, so we can write

$$\forall\,(\lambda,\ \lambda') \in (\overline{I_0}\cap\overline{I_k^n})^2,\ |\phi_k(\lambda') - \phi_k(\lambda)| \le M_k\|f\|_{H_\varepsilon^s}\|g\|_{H_\varepsilon^s}|p'-p|^{\theta/N_j^n}\,, \tag{3.49}$$

where the constant M_k depends neither on f nor g. In view of extending (3.49) to any λ and λ' in $\overline{I_0}$, two cases have to be distinguished now.

- First case: λ_n is monotone on $]p_{j-1}^n, p_j^n[\ \cup\ ]p_j^n, p_{j+1}^n[$. Thus, $\overline{I_{j-1}^n}\cap\overline{I_j^n} = \{\tau\}$ and, according to (3.49), the function

$$\phi(\lambda) = \begin{cases} \phi_{j-1}^n(\lambda) & \text{if } \lambda \in \overline{I_{j-1}^n}\cap\overline{I_0}\\ \phi_j^n(\lambda) & \text{if } \lambda \in \overline{I_j^n}\cap\overline{I_0}\,, \end{cases}$$

satisfies

$$\forall\ (\lambda,\ \lambda') \in\ \overline{I_0}^2,\ |\phi(\lambda') - \phi(\lambda)| \leq a_j^n \|f\|_{H_\varepsilon^s} \|g\|_{H_\varepsilon^s} |\lambda' - \lambda|^{\theta/N_j^n}\ , \tag{3.50}$$

for some independent constant a_j^n of f and g. Moreover, ψ vanishes at the ends of $\overline{P_0}$, so ϕ vanishes at $\overline{I_0} \setminus I_0$.

- Second case: λ_n is not monotone on $]p_{j-1}^n, p_j^n[\ \cup\ ]p_j^n, p_{j+1}^n[$. As I_{j-1}^n and I_j^n have symmetric roles, we can assume for example that $I_{j-1}^n \subset I_j^n$. Then, by setting

$$\phi(\lambda) = \begin{cases} \phi_{j-1}^n(\lambda) + \phi_j^n(\lambda) & \text{if } \lambda \in \overline{I_{j-1}^n}\ \cap\ \overline{I_0} \\ \phi_j^n(\lambda) & \text{if } \lambda \in \overline{(I_j^n \backslash I_{j-1}^n)}\ \cap\ \overline{I_0}, \end{cases}$$

it can be verified that ϕ satisfies (3.50) and vanishes at $\overline{I_0} \backslash I_0$ again.

Now, as $R_n^{j,0}(z)$ reduces to $\int_{\overline{I_0}} \frac{\phi(\lambda)}{\lambda - z} d\lambda$ for any $z \in K \cap \mathbb{C}^{\pm}$, Lemma 3.1 finally involves:

$$\forall\ (z, z') \in (K \cap \mathbb{C}^{\pm})^2, \quad |R_n^{j,0}(z') - R_n^{j,0}(z)| \leq a_j^n \|f\|_{H_\varepsilon^s} \|g\|_{H_\varepsilon^s} |z' - z|^{\theta/N_j^n}\ . \tag{3.51}$$

Moreover, we remark that the condition $\phi(\tau) = 0$ warrants

$$\lim_{z \to \tau, \pm \mathrm{Im}(z) > 0} R_n^{j,0}(z) = p.v. \left(\int_{\mathbb{R}} \frac{\beta_0(p) h_n(p)}{\lambda_n(p) - \tau} dp \right)\ . \tag{3.52}$$

Next, $\mathrm{supp}(\beta_1) \subset \mathbb{R} \backslash \bar{P}$, so (3.46) proves that $R_n^{j,1}$ is Lipschitz continuous on $K \cap \mathbb{C}^{\pm}$. As $\theta / N_j^n \leq 1$, it can finally be deduced from (3.51) that

$$\forall\ (z, z') \in (K \cap \mathbb{C}^{\pm})^2, \quad |R_n^j(z') - R_n^j(z)| \leq C_n^j \|f\|_{H_\varepsilon^s} \|g\|_{H_\varepsilon^s} |z' - z|^{\theta/N_j^n}\ , \tag{3.53}$$

for some constant C_n^j independent of f and g.

Now, as $J_n'(\tau)$ is at most finite, (3.32) is verified when any R_n^j, $j \in J_n'(\tau)$, satisfies (3.53). This condition can be fulfilled by taking functions f and g in a suitable intersection of the previous NH_ε^s-spaces. Set also $Z_\tau = \{(n,j),\ n \in \mathbb{N}^*,\ j \in J_n'(\tau)\}$. Then, for any multi-indice $\underline{a} = \left(a_j^n\right)_{(n,j) \in Z_\tau} \in (\mathbb{N}^*)^{Z_\tau}$, whose modulus is $|\underline{a}| = \max_{(n,j) \in Z_\tau} |a_j^n|$, and for any $s > |\underline{a}| - 1/2$, we have already remarked that $NH_\varepsilon^s(n,j,a_j^n)$ is a closed subset of H_ε^s. So the intersection

$$NH_\varepsilon^s(\tau, \underline{a}) = \bigcap_{(n,j) \in Z_\tau} NH_\varepsilon^s(n,j,a_j^n)$$

is also closed. Now, it only remains to find a suitable multi-indice $\underline{a}$ for which a limiting absorption principle can be deduced in the space $NH_\varepsilon^s(\tau, \underline{a})$.

3.4.4. *A limiting absorption principle for M at τ*

Assume now that a_j^n belongs to $\{1, 2, \ldots, N_j^n\}$ for any (n,j) in Z_τ, and set $\underline{b} = \left(N_j^n - a_j^n\right)_{(n,j) \in Z_K}$. Next, take $s' > |\underline{b}| - 1/2$ such that $s + s' > |\underline{N}|$. At last, set

$t = \min(1, s - |\underline{a}| + 1/2)$, $t' = \min(1, s' - |\underline{b}| + 1/2)$ and $\theta \in [0, \frac{t+t'-1}{2}[$. Thus, recalling (3.36), (3.37), (3.43), (3.45) and (3.53), we finally have proved that for any $(z, z') \in (K \cap \mathbb{C}^{\pm})^2$,

$$|(R_M^{\pm}(z')f, g)_{H_\varepsilon^s} - (R_M^{\pm}(z)f, g)_{H_\varepsilon^s}| \leq C_M \|f\|_{H_\varepsilon^s} \|g\|_{H_\varepsilon^s} |z - z'|^{\theta/|\underline{N}|},$$

where the constant C_M is independent of f and g. Thus, as shown in Sec. 3.1, the general extension theorem of uniformly continuous functions involves:

Theorem 3.2. *Let $\tau \in \mathcal{T}$ and K be a compact subset of $(\mathbb{C} \backslash \mathcal{T}) \cup \{\tau\}$. Next, for any multi-indice $\underline{a} = (a_j^n)_{(n,j) \in Z_K}$ such that $a_j^n \in \{1, 2, \ldots, N_j^n\}$ for any (n, j) in Z_K, set $\underline{b} = (N_j^n - a_j^n)_{(n,j) \in Z_K}$. At last, take $s > |\underline{a}| - 1/2$* and *$s' > |\underline{b}| - 1/2$ such that $s + s' > |\underline{N}|$.*

Thus, the function $z \mapsto R_M(z)$ extends in a $\theta/|\underline{N}|$-Hölder continuous function on $K \cap \overline{\mathbb{C}^{\pm}}$ for any $\theta \in [0, t + t' - 1/2[$, where $t = \min(1, s - |\underline{a}| + 1/2)$ and $t' = \min(1, s' - |\underline{b}| + 1/2)$. Indeed, there is a constant $c = c(s, s', \theta, K)$ such that

$$\forall\, z, z' \in K \cap \overline{\mathbb{C}^{\pm}},\ \|R_M^{\pm}(z') - R_M^{\pm}(z)\|_{B(NH_\varepsilon^s(\tau,\underline{a}),(NH_\varepsilon^{s'}(\tau,\underline{b}))')} \leq c|z' - z|^{\theta/(|\underline{N}|+1)}.$$

Finally, under the assumptions of Theorem 3.2, we deduce from (3.52) and Lemma 3.1 that:

$$\lim_{z \to \tau, \pm \mathrm{Im}(z) > 0} (R_M^{\pm}(z)f, g)_{H_\varepsilon}$$
$$= \sum_{n \geq 1} \left\{ p.v. \left(\int_{\mathbb{R}} \frac{h_n(p)}{\lambda_n(p) - \tau} dp \right) \pm i\pi \sum_{q \in \lambda_n^{-1}(\{\tau\}), \lambda_n'(q) \neq 0} \frac{h_n(q)}{|\lambda_n'(q)|} \right\}.$$

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