LIMITING ABSORPTION PRINCIPLE FOR THE MAGNETIC DIRICHLET LAPLACIAN IN A HALF-PLANE

NICOLAS POPOFF AND ERIC SOCCORSI

ABSTRACT. We consider the Dirichlet Laplacian in the half-plane with constant magnetic field. Due to the translational invariance, this operator admits a fiber decomposition and a family of dispersion curves, that are real analytic functions. Each of them is simple and monotonically decreasing from positive infinity to a finite value, which is the corresponding Landau level. These finite limits are thresholds in the purely absolutely continuous spectrum of the magnetic Laplacian. We prove a limiting absorption principle for this operator, both outside and at the thresholds. Finally, we establish analytic and decay properties for functions lying in the absorption spaces. We point out that the analysis carried out in this paper is rather general, and can be adapted to a wide class of fibered magnetic Laplacians with thresholds in their spectrum, that are finite limits of their band functions.

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1. INTRODUCTION

In the present article we consider the Hamiltonian $(-i\nabla+A)^2$ with magnetic potential $A(x,y) := (0,x)$, defined in the half-plane $\Omega := \{(x,y) \in \mathbb{R}^2,\ x > 0\}$. We impose Dirichlet boundary conditions at $x = 0$ and introduce the self-adjoint realization

$$H := -\partial_x^2 + (-i\partial_y - x)^2,$$

initially defined in $C_0^\infty(\Omega)$ and then closed in $L^2(\Omega)$. This operator models the planar motion of a quantum charged particle (the various physical constants are taken equal to 1) constrained to Ω and submitted to an orthogonal magnetic field of strength $\operatorname{curl}\ A = 1$. This Hamiltonian has already been studied in several articles (see e.g. [8, 15, 6, 16, 20]).

The Schrödinger operator H is translationally invariant in the y-direction and admits a fiber decomposition with fiber operators which have purely discrete spectrum. The corresponding dispersion curves (also named band functions in this text) are real analytic functions in $\mathbb{R}$, monotonically decreasing from positive infinity to the n-th Landau level $E_n := 2n-1$ for $n \in \mathbb{N}^*$. As a consequence, the spectrum of H is absolutely continuous and equals the interval $[E_1, +\infty)$. Hence the resolvent operator $R(z) := (H-z)^{-1} \in \mathcal{B}(L^2(\Omega))$ depends analytically on z in $\mathbb{C} \setminus [E_1, +\infty)$ and $R^\pm(z) := R(z)$ is well defined for every $z \in \mathbb{C}^\pm := \{z \in \mathbb{C},\ \pm\Im z > 0\}$.

Since H has a continuous spectrum, the spectral projector $E(a,b)$ of H, associated with the interval (a,b), $a<b$, expresses as

$$E(a,b)=\frac{1}{2i\pi}\lim_{\varepsilon\downarrow 0}\int_a^b \left(R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\right)\mathrm{d}\lambda,$$

by the spectral theorem. Suitable functions of the operator H may therefore be expressed in terms of the limits of the resolvent operators $\lim_{\varepsilon\downarrow 0}R(\lambda\pm i\varepsilon)$ for $\lambda\in\mathbb{R}$. As a matter of fact the Schrödinger propagator e^{-itH} associated with H reads

$$e^{-itH}=\frac{1}{2i\pi}\lim_{\varepsilon\downarrow 0}\int_{E_1}^{+\infty} e^{-it\lambda}\left(R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\right)\mathrm{d}\lambda,\ t>0.$$

This motivates for a quantitative version of the convergence $\lim_{\varepsilon\downarrow 0}R(\lambda\pm i\varepsilon)$, for $\lambda\in[E_1,+\infty)$, known as the *limiting absorption principle* (abbreviated to LAP in the sequel). Notice moreover that a LAP is a useful tool for the analysis of the scattering properties of H, and more specifically for the proof of the existence and the completeness of the wave operators (see e.g. [25, Section XI]). The main purpose of this article is to establish a LAP for H. That is, for each $\lambda\in[E_1,+\infty)$, we aim to prove that $R(\lambda\pm i\varepsilon)$ has a limit as $\varepsilon\downarrow 0$, in a suitable sense we shall make precise further.

There is actually a wide mathematical literature on LAP available for various operators of mathematical physics (see e.g. [29, 10, 19, 1, 28, 27, 3, 9]). More specifically, the case of analytically fibered self-adjoint operators was addressed in e.g. [7, 13, 26]. Such an operator A is unitarily equivalent to the multiplier by a family of real analytic dispersion curves, so its spectrum is the closure of the range of its band functions. Generically, energies associated with a "flat" of any of the band functions $\{\lambda_n,\ n\in\mathbb{N}^*\}$, are *thresholds* in the spectrum of A. More precisely, a threshold of the operator A is any real number λ satisfying $\inf_U|\lambda_n'\circ\lambda_n^{-1}|=0$ for some $n\in\mathbb{N}^*$ and all neighborhoods U of λ in $\overline{\lambda_n(\mathbb{R})}$. We denote by $\mathcal{T}$ the set of thresholds.

The occurrence of a LAP outside the thresholds of analytically fibered operators is a rather standard result. It is tied to the existence of a Mourre inequality at the prescribed energies (see [13, 12]), arising from the non-zero velocity of the dispersion curves for the corresponding frequencies. More precisely, given an arbitrary compact subset $K\subset\mathbb{C}\setminus\mathcal{T}$, we shall extend $z\mapsto R^{\pm}(z)$ to a Hölder continuous function on $K\cap\overline{\mathbb{C}^{\pm}}$ in the norm-topology of $\mathcal{B}(L^{2,s}(\Omega),L^{2,-s}(\Omega))$ for any $s\in(1/2,+\infty)$. Here and henceforth, the Hilbert space

$$L^{2,\sigma}(\Omega):=\{u:\Omega\to\mathbb{C}\text{ measurable},\ (x,y)\mapsto(1+y^2)^{\sigma/2}u(x,y)\in L^2(\Omega)\},$$

is endowed with the scalar product $\langle u,v\rangle_{L^{2,\sigma}(\Omega)}:=\int_\Omega(1+y^2)^\sigma u(x,y)\overline{v(x,y)}\mathrm{d}x\mathrm{d}y$.

Evidently, local extrema of the dispersion curves are thresholds in the spectrum of fibered operators. Any such energy being a critical point of some band function, it is referred to as an *attained threshold*. Actually, numerous operators of mathematical physics modeling the propagation of acoustic, elastic, or electromagnetic waves in stratified media [7, 4, 5, 26], and various magnetic Hamiltonians [11, 15, 30], have all their thresholds among local minima of their band functions. It is rather well known that we can derive a LAP at an attained threshold upon imposing a suitable vanishing condition (depending on the level of degeneracy of the critical point) on the Fourier transform of functions in $L^{2,\sigma}(\Omega)$, at the corresponding frequency. We refer to [4, 26] for the analysis of this problem in the general case.

Nevertheless, none of the above mentioned papers is relevant for the operator H studied in this paper. This comes from the unusual behavior of the band functions of H at positive infinity. Actually, there exists a countable set $\mathcal{T}$ of thresholds $\{E_n,\ n \in \mathbb{N}^*\}$ in the spectrum of H, but in contrast to the framework examined in [4, 26], none of these thresholds are attained. This particular behavior raises several technical problems in the derivation of a LAP for H at E_n, $n \in \mathbb{N}^*$. The main purpose of this paper is to establish a LAP for H in any arbitrary compact subset $K \subset \mathbb{C}$ (which may contain one or several thresholds E_n, $n \in \mathbb{N}^*$). The corresponding result is given in Theorem 2.6. It is stated for the topology of the norm in $\mathcal{B}(X_K, (X_K)')$, where X_K is a suitable subspace of $L^{2,\sigma}(\Omega)$ associated with $\sigma > 1/2$. The linear space X_K is dense in $L^2(\Omega)$ and is made of $L^{2,\sigma}(\Omega)$-functions with smooth Fourier coefficients vanishing suitably at every threshold of H lying in K. Otherwise stated there is an actual LAP at E_n, $n \in \mathbb{N}^*$, even though E_n is a *non attained threshold* of H. We point out that the method used in the derivation of the LAP for H is quite general and may be generalized to a wide class of fibered operators (such as the ones examined in [15, 30, 6, 17, 23]) with non attained thresholds in their spectrum.

Moreover, it turns out that the functions of X_K exhibit interesting geometrical properties. Namely, if $E_n \in K$, then the asymptotic behavior of the n-th band function of H at positive infinity, described in [16, Theorem 1.4], translates into super-exponential decay in the x-variable (orthogonal to the edge) of their n-th harmonic. Such a behavior, expressed in Theorem 3.5, is typical of magnetic Laplacians, as explained in Remark 3.7.

1.1. Spectral decomposition associated with the model.

In this subsection we collect several basic properties of the fiber decomposition of the operator H that are needed in the derivation of Theorems 2.6 and 3.5.

The Schrödinger operator H is translationally invariant in the longitudinal direction y and therefore allows a direct integral decomposition

$$\mathcal{F}_y^* H \mathcal{F}_y = \int_{k\in\mathbb{R}}^{\oplus} \mathfrak{h}(k)\mathrm{d}k, \tag{1}$$

where $\mathcal{F}_y$ denotes the partial Fourier transform with respect to y, and the fiber operator $\mathfrak{h}(k) := -\partial_x^2 + (x-k)^2$ acts in $L^2(\mathbb{R}_+^*)$ with a Dirichlet boundary condition at $x = 0$. Since the effective potential $(x-k)^2$ is unbounded as x goes to infinity, each $\mathfrak{h}(k)$, $k \in \mathbb{R}$, has a compact resolvent, and hence a purely discrete spectrum. We denote by $\{\lambda_n(k),\ n \in \mathbb{N}^*\}$ the non-decreasing sequence of the eigenvalues of $\mathfrak{h}(k)$, each of them being simple. Furthermore, for $k \in \mathbb{R}$, we introduce a family $\{u_n(\cdot, k),\ n \in \mathbb{N}^*\}$ of eigenfunctions of the operator $\mathfrak{h}(k)$, which satisfy

$$\mathfrak{h}(k)u_n(x,k) = \lambda_n(k)u_n(x,k),\ x \in \mathbb{R}_+^*,$$

and form an orthonormal basis in $L^2(\mathbb{R}_+^*)$.

As $\{\mathfrak{h}(k),\ k \in \mathbb{R}\}$ is a Kato analytic family, the functions $\mathbb{R} \ni k \mapsto \lambda_n(k) \in (0, +\infty)$, $n \in \mathbb{N}^*$, are analytic (see e.g. [24, Theorem XII.12]). Moreover, they are monotonically decreasing in $\mathbb{R}$, according to [8, Lemma 2.1 (ii)], and the Max-Min principle yields (see [8, Lemma 2.1 (iii) and (v)]) that

$$\lim_{k\to-\infty}\lambda_n(k)=+\infty \text{ and } \lim_{k\to+\infty}\lambda_n(k)=E_n,\ n\in\mathbb{N}^*.$$

As a consequence, the general theory of fibered operators (see e.g. [24, Section XIII.16]) implies that the spectrum of H is purely absolutely continuous, with

$$\sigma(H)=\overline{\cup_{n\in\mathbb{N}^*}\lambda_n(\mathbb{R})}=[E_1,+\infty).$$

For all $n\in\mathbb{N}^*$ we define the n^{th} Fourier coefficient $f_n\in L^2(\mathbb{R})$ of $f\in L^2(\Omega)$ by

$$f_n(k):=\langle\mathcal{F}_y f(\cdot,k),u_n(\cdot,k)\rangle_{L^2(\mathbb{R}_+)},\ k\in\mathbb{R},$$

and introduce its n^{th} harmonic as

$$\Pi_n f(x,y):=\int_{\mathbb{R}} e^{iky}f_n(k)u_n(x,k)\mathrm{d}k,\ (x,y)\in\Omega. \tag{2}$$

In view of (1), we have the standard Fourier decomposition $f=\sum_{n\in\mathbb{N}^*}\Pi_n f$ in $L^2(\Omega)$ and the following Parseval identity

$$\|f\|^2_{L^2(\Omega)}=\sum_{n\in\mathbb{N}^*}\|f_n\|^2_{L^2(\mathbb{R})}, \tag{3}$$

involving that the linear mapping $\pi_n : f\mapsto f_n$ is continuous from $L^2(\Omega)$ into $L^2(\mathbb{R})$ for each $n\in\mathbb{N}^*$. Let us now recall from [7, Proposition 3.2] the following useful properties of the restriction of π_n to $L^{2,s}(\Omega)$ for $s\in(1/2,+\infty)$.

Lemma 1.1. *Fix $s>1/2$. Then the operators π_n are uniformly bounded with respect to $n\in\mathbb{N}^*$ from $L^{2,s}(\Omega)$ into $L^\infty(\mathbb{R})$:*

$$\exists C(s)>0,\ \forall n\in\mathbb{N}^*,\ \forall f\in L^{2,s}(\Omega),\ \forall k\in\mathbb{R},\ |f_n(k)|\le C(s)\|f\|_{L^{2,s}(\Omega)}. \tag{4}$$

Moreover for any $\alpha\in[0,\min(1,s-1/2))$, each operator π_n, $n\in\mathbb{N}^*$, is bounded from $L^{2,s}(\Omega)$ into $\mathcal{C}^{0,\alpha}_{\text{loc}}(\mathbb{R})$, the set of locally Hölder continuous functions in $\mathbb{R}$, of exponent α. Namely, there exists a function $C_{n,\alpha,s}\in\mathcal{C}^0(\mathbb{R}^2;\mathbb{R}_+)$, such that we have

$$\forall f\in L^{2,s}(\Omega),\ \forall(k,k')\in\mathbb{R}^2,\ |f_n(k')-f_n(k)|\le C_{n,\alpha,s}(k,k')\|f\|_{L^{2,s}(\Omega)}|k'-k|^\alpha. \tag{5}$$

2. Limiting absorption principle

For all $z\in\mathbb{C}\setminus[E_1,+\infty)$ and $f,g\in L^2(\Omega)$, the standard functional calculus yields

$$\langle R(z)f,g\rangle_{L^2(\Omega)}=\sum_{n\ge1}r_n(z),\quad\text{with}\quad r_n(z):=\int_{\mathbb{R}}\frac{f_n(k)\overline{g_n(k)}}{\lambda_n(k)-z}\mathrm{d}k,\ n\in\mathbb{N}^*. \tag{6}$$

Since $I_n:=\lambda_n(\mathbb{R})=(E_n,+\infty)$ for each $n\in\mathbb{N}^*$, the function $z\mapsto r_n(z)$ is analytic on $\mathbb{C}\setminus\overline{I}_n$, so $r_n^\pm : z\in\mathbb{C}^\pm\mapsto r_n(z)$ is well defined. In light of (6), it suffices that each $r_n^\pm$, $n\in\mathbb{N}^*$, be suitably extended to some locally Hölder continuous function in $\overline{\mathbb{C}^\pm}$, in order to obtain a LAP for the operator H.

2.1. **Singular Cauchy integrals.** Let $n \in \mathbb{N}^*$ be fixed. Bearing in mind that λ_n is an analytic diffeomorphism from $\mathbb{R}$ onto I_n, we denote by λ_n^{-1} the function inverse to λ_n and put $\tilde{\psi}_n(\lambda) := (\psi \circ \lambda_n^{-1})(\lambda)$ for any function $\psi : \mathbb{R} \to \mathbb{R}$. Then, upon performing the change of variable $\lambda = \lambda_n(k)$ in the integral appearing in (6), we get for every $z \in \mathbb{C} \setminus \overline{I_n}$ that

$$r_n(z) = \int_{I_n} \frac{H_n(\lambda)}{\lambda - z} \mathrm{d}\lambda \text{ with } H_n := \frac{\tilde{f}_n \, \overline{\tilde{g}_n}}{\tilde{\lambda}_n'} = \frac{(f_n \circ \lambda_n^{-1})\overline{(g_n \circ \lambda_n^{-1})}}{\lambda_n' \circ \lambda_n^{-1}}. \tag{7}$$

Therefore, the Cauchy integral $r_n(z)$ is singular for $z \in \overline{I}_n = [E_n, +\infty)$. Our main tool for extending singular Cauchy integrals of this type to locally Hölder continuous functions in $\overline{\mathbb{C}^\pm}$ is the Plemelj-Privalov Theorem (see e.g. [22, Section 2.22]), stated below.

Lemma 2.1. *Let $\alpha \in (0,1]$ and let $\psi \in \mathcal{C}^{0,\alpha}(\overline{I})$, where $I := (a,b)$, $a < b$, is an open bounded subinterval of $\mathbb{R}$. Then the mapping $z \mapsto r(z) := \int_I \frac{\psi(t)}{t-z} \mathrm{d}t$, defined in $\overline{\mathbb{C}^\pm} \setminus \overline{I}$, satisfies for every $\lambda \in I$:*

$$\lim_{\varepsilon \downarrow 0} r(\lambda \pm i\varepsilon) = r^\pm(\lambda) := p.v. \left(\int_I \frac{\psi(t)}{t - \lambda} \mathrm{d}t \right) \pm i\pi\psi(\lambda).$$

Moreover the function

$$r^\pm(\lambda) := \begin{cases} r(\lambda) & \text{if } \lambda \in \overline{\mathbb{C}^\pm} \setminus \overline{I}, \\ r^\pm(\lambda) & \text{if } \lambda \in I, \end{cases}$$

is analytic in $\mathbb{C}^\pm$ and locally Hölder continuous of order α in $\overline{\mathbb{C}^\pm} \setminus \{a,b\}$, in the sense that there exists $C_{I,\alpha} \in \mathcal{C}^0((\overline{\mathbb{C}^\pm} \setminus \{a,b\})^2; \mathbb{R}_+)$, such that we have

$$\forall z, z' \in \overline{\mathbb{C}^\pm} \setminus \{a,b\}, \ |r^\pm(z') - r^\pm(z)| \le \|\psi\|_{C^{0,\alpha}(\overline{I})} C_{I,\alpha}(z,z') |z' - z|^\alpha.$$

In addition, if $\psi(a) = \psi(b) = 0$, then $r^\pm$ extends to a locally Hölder continuous function of order α in $\overline{\mathbb{C}^\pm}$.

2.2. **Limiting absorption principle outside the thresholds.** In this subsection we establish a LAP for H outside its thresholds $\mathcal{T} = \{E_n, \ n \in \mathbb{N}^*\}$. This is a rather standard result that we state here for the convenience of the reader. For the sake of completeness we also recall its proof, which requires several ingredients that are useful for the derivation of Theorem 2.6, given in Subsection 2.3.

Proposition 2.2. *Let K be a compact subset of $\mathbb{C} \setminus \mathcal{T}$. Then for all $s \in (1/2, +\infty)$ and any $\alpha \in [0, \min(1, s - 1/2))$, the resolvent $z \mapsto R^\pm(z)$ extends to $K \cap \overline{\mathbb{C}^\pm}$ in a Hölder continuous function of order α, still denoted by $R^\pm$, for the topology of the norm in $\mathcal{B}(L^{2,s}(\Omega), L^{2,-s}(\Omega))$. Namely, there exists a constant $C = C(K, s, \alpha) > 0$, such that the estimate*

$$\|(R^\pm(z') - R^\pm(z))f\|_{L^{2,-s}(\Omega)} \le C \|f\|_{L^{2,s}(\Omega)} |z' - z|^\alpha$$

holds for all $f \in L^{2,s}(\Omega)$ and all $z, z' \in K \cap \overline{\mathbb{C}^\pm}$.

Proof. Let f and g be in $L^{2,s}(\Omega)$. We use the notations introduced in (6)-(7). Since K is bounded, then there exists $N = N(K) \in \mathbb{N}^*$ such that $d_K := \inf_{z \in K}(E_N - \mathrm{Re}(z)) > 0$. In light of (3), this entails through straightforward computations that $z \mapsto \sum_{m \geq N} r_m(z)$ is Lipschitz continuous in K, with

$$\forall z \in K, \ \left\| \sum_{m \geq N} r_m(z) \right\|_{\mathcal{C}^{0,1}(K)} \leq d_K^{-2} \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \tag{8}$$

Therefore, it suffices to examine each r_m, for $m = 1, \dots, N-1$, separately. Taking into account that K is a compact subset of $\mathbb{C} \setminus \mathcal{T}$, we pick an open bounded subinterval $I = (a, b)$, with $E_m < a < b$, such that $K \cap I_m \subset I$. With reference to (7), we have the following decomposition for each $z \in K \cap \mathbb{C}^{\pm}$:

$$r_m(z) = r_m(z; I) + r_m(z; I_m \setminus \overline{I}) \text{ where } r_m(z, J) := \int_J \frac{H_m(z)}{\lambda - z} \mathrm{d}\lambda \text{ for any } J \subset I_m. \tag{9}$$

As $E_m \notin \overline{I}$, then λ_m^{-1} and $\lambda_m' \circ \lambda_m^{-1}$ are both Lipschitz continuous in $\overline{I}$, and it holds true that $\inf_{\lambda \in \overline{I}} |\lambda_m' \circ \lambda_m^{-1}(\lambda)| > 0$. Thus, we deduce from Lemma 1.1 that $H_m \in \mathcal{C}^{0,\alpha}(\overline{I})$ verifies

$$\|H_m\|_{\mathcal{C}^{0,\alpha}(\overline{I})} \leq c_m \|f\|_{L^{2,s}(\Omega)} \|g\|_{L^{2,s}(\Omega)}, \tag{10}$$

for some constant $c_m > 0$, independent of f and g. From this and Lemma 2.1, it then follows that $r_m(\cdot; I)$ extends to a locally Hölder continuous function of order α in $K \cap \overline{\mathbb{C}^{\pm}}$, obeying

$$\|r_m(\cdot; I)\|_{\mathcal{C}^{0,\alpha}(K \cap \overline{\mathbb{C}^{\pm}})} \leq C_m \|H_m\|_{\mathcal{C}^{0,\alpha}(\overline{I})}, \tag{11}$$

where the positive constant C_m depends neither on f nor on g.

Next, the Euclidean distance between $I_m \setminus \overline{I}$ and $K \cap I_m$ being positive, from the very definition of I, we get $\delta_K(I) := \inf\{|\lambda - z|, \ \lambda \in I_m \setminus \overline{I}, \ z \in K \cap \overline{\mathbb{C}^{\pm}}\} > 0$, by the compactness of K. Therefore $z \mapsto r_m(z; I_m \setminus \overline{I})$ is Lipschitz continuous in K and satisfies

$$\|r_m(\cdot; I_m \setminus \overline{I})\|_{\mathcal{C}^{0,1}(K)} \leq \delta_K(I)^{-2} \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}, \tag{12}$$

according to (3).

Finally, putting (8)-(9) and (10)-(12) together, and recalling that the injection $L^{2,s}(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, we end up getting a constant $C > 0$, such that the estimate

$$|\langle R^{\pm}(z) f, g \rangle_{L^2(\Omega)} - \langle R^{\pm}(z') f, g \rangle_{L^2(\Omega)}| \leq C \|f\|_{L^{2,s}(\Omega)} \|g\|_{L^{2,s}(\Omega)} |z - z'|^{\alpha},$$

holds uniformly in $f, g \in L^{2,s}(\Omega)$ and $z, z' \in K \cap \overline{\mathbb{C}^{\pm}}$. The result follows from this and the fact that $L^{2,-s}(\Omega)$ and the space $\mathcal{B}(L^{2,s}(\Omega), \mathbb{C})$ of continuous linear forms on $L^{2,s}(\Omega)$ are isometric, with the duality pairing

$$\forall f \in L^{2,-s}(\Omega), \ \forall g \in L^{2,s}(\Omega), \ \langle f, g \rangle_{L^{2,-s}(\Omega), L^{2,s}(\Omega)} := \int_{\Omega} f(x,y) \overline{g(x,y)} \mathrm{d}x \mathrm{d}y.$$

□

2.3. **Limiting absorption principle at the thresholds.** We now examine the case of a compact subset $K \subset \mathbb{C}$ containing one or several thresholds of H, i.e. such that $K \cap \mathcal{T} \neq \emptyset$. Since $\lim_{n\to+\infty} E_n = +\infty$ then the bounded set K contains at most a finite number of thresholds. For the sake of clarity, we first investigate the case where K contains exactly one threshold:

$$\exists n \in \mathbb{N}^*,\ K \cap \mathcal{T} = \{E_n\}. \tag{13}$$

The target here is the same as in Subsection 2.2, that is to establish a LAP for H in K. Actually, for all $s \in (1/2, +\infty)$ and any $\alpha \in [0, \min(s-1/2, 1))$, it is clear from the proof of Proposition 2.2 that $z \mapsto \sum_{m\neq n} r_m(z)$ can be regarded as a α-Hölder continuous function in $K \cap \overline{\mathbb{C}^{\pm}}$ with values in $\mathcal{B}(L^{2,s}(\Omega), L^{2,-s}(\Omega))$.

Thus we are left with the task of suitably extending $z \mapsto r_n(z)$ in $K \cap \overline{\mathbb{C}^{\pm}}$. But, since $H_n(\lambda)$ may actually blow up as λ tends to E_n, it is apparent that the method used in the proof of Proposition 2.2 does not apply to $r_n(z)$ when z lies in the vicinity of E_n. This is due to the vanishing of the denominator $\lambda_n' \circ \lambda_n^{-1}(\lambda)$ of $H_n(\lambda)$ in (7) as λ approaches E_n, or, equivalently, to the flattening of $\lambda_n'(k)$ when k goes to $+\infty$. We shall compensate this particular asymptotic behavior of the dispersion curve $k \mapsto \lambda_n(k)$ by imposing appropriate conditions on the functions f and g, so that the numerator $k \mapsto f_n(k)\overline{g_n(k)}$ decays sufficiently fast at $+\infty$. This requires that the following functional spaces be preliminarily introduced.

- *Functional spaces.* For any open subset $J \subset I_n = (E_n, +\infty)$ and the non vanishing function

$$\mu_n := |\lambda_n' \circ \lambda_n^{-1}|^{-1/2}, \tag{14}$$

on J, we denote by $\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{J}) := \{\psi,\ \mu_n\psi \in \mathcal{C}^{0,\alpha}(\overline{J})\}$ (resp. $L^2_{\mu_n}(J) := \{\psi,\ \mu_n\psi \in L^2(J)\}$), the μ_n-weighted space of Hölder continuous functions of order $\alpha \in [0,1)$ (resp., square integrable functions) in J. Endowed with the norm $\|\psi\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{J})} := \|\mu_n\psi\|_{\mathcal{C}^{0,\alpha}(\overline{J})}$ (resp., $\|\psi\|_{L^2_{\mu_n}(J)} := \|\mu_n\psi\|_{L^2(J)}$), $\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{J})$ (resp., $L^2_{\mu_n}(J)$) is a Banach space since this is the case for $\mathcal{C}^{0,\alpha}(\overline{J})$ (resp., $L^2(J)$).

Remark 2.3. Since μ_n is not defined at E_n then it is understood in the particular case where $E_n \in \overline{J}$, that $\psi \in \mathcal{C}^{0,\alpha}_{\mu_n}(\overline{J})$ if and only if $\mu_n\psi$ extends continuously to a function lying in $C^{0,\alpha}(\overline{J})$.

The above definitions translate through the linear isometry $\Lambda_n : f \mapsto f \circ \lambda_n^{-1}$ from $L^2(\mathbb{R})$ into $L^2_{\mu_n}(I_n)$, to

$$\mathcal{K}^\alpha_n(\mathbb{R}) = \Lambda_n^{-1}(\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n) \cap L^2_{\mu_n}(I_n)) := \{f \in L^2(\mathbb{R}),\ \Lambda_n f \in \mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)\},$$

which is evidently a Banach space for the norm $\|f\|_{\mathcal{K}^\alpha_n(\mathbb{R})} := \|\Lambda_n f\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)} + \|\Lambda_n f\|_{L^2_{\mu_n}(I_n)}$. As a consequence the set

$$\mathcal{X}^{s,\alpha}_n(\Omega) = \pi_n^{-1}(\mathcal{K}^\alpha_n(\mathbb{R})) \cap L^{2,s}(\Omega) := \{f \in L^{2,s}(\Omega),\ \pi_n f \in \mathcal{K}^\alpha_n(\mathbb{R})\},\ s \in \mathbb{R}^*_+,$$

equipped with its natural norm $\|f\|_{\mathcal{X}^{s,\alpha}_n(\Omega)} := \|\pi_n f\|_{\mathcal{K}^\alpha_n(\mathbb{R})} + \|f\|_{L^{2,s}(\Omega)}$ is a Banach space as well.

On $\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)$ we define the linear form $\delta_{E_n} : \psi \mapsto (\mu_n\psi)(E_n)$. Notice from the embedding $\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n) \subset \mathcal{C}^0_{\mu_n}(\overline{I}_n) := \{\psi,\ \mu_n\psi \in \mathcal{C}^0(\overline{I}_n)\}$, that δ_{E_n} is well defined since $I_n \ni \lambda \mapsto (\mu_n\psi)(\lambda)$

extends to a continuous function in $\overline{I}_n$. Furthermore, we have $|\delta_{E_n}(\psi)| \leq \|\psi\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)}$ for any $\psi \in \mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)$ so the linear form δ_{E_n} is continuous on $\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)$. Let us now introduce the subspace

$$\begin{aligned} (15) \qquad \mathcal{X}^{s,\alpha}_{n,0}(\Omega) &:= \mathcal{X}^{s,\alpha}_{n}(\Omega) \cap (\Lambda_n \pi_n)^{-1}(\ker \delta_{E_n}) \\ &= \{ f \in L^{2,s}(\Omega),\ \mu_n \tilde{f}_n \in \mathcal{C}^{0,\alpha}(\overline{I}_n) \cap L^2(I_n) \text{ and } (\mu_n \tilde{f}_n)(E_n) = 0 \}, \end{aligned}$$

where, as above, $\tilde{f}_n$ stands for $\Lambda_n \pi_n f$. Since δ_{E_n} is continuous then $\mathcal{X}^{s,\alpha}_{n,0}(\Omega)$ is closed in $\mathcal{X}^{s,\alpha}_{n}(\Omega)$. Therefore it is a Banach space for the norm

$$\|f\|_{\mathcal{X}^{s,\alpha}_n(\Omega)} = \|\tilde{f}_n\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)} + \|\tilde{f}_n\|_{L^2_{\mu_n}(I_n)} + \|f\|_{L^{2,s}(\Omega)}.$$

Moreover, $\mathcal{C}^\infty_0(\mathbb{R})$ being dense in $L^2(\mathbb{R})$, we deduce from the imbedding $\pi_n^{-1}(\mathcal{C}^\infty_0(\mathbb{R})) \subset \mathcal{X}^{s,\alpha}_{n,0}(\Omega)$ that $\mathcal{X}^{s,\alpha}_{n,0}(\Omega)$ is dense in $L^2(\Omega)$ endowed with its usual norm-topology.

Summing up, we have obtained the:

Lemma 2.4. *The set $\mathcal{X}^{s,\alpha}_{n,0}(\Omega)$, defined in (15), is a Banach space that is dense in $L^2(\Omega)$.*

• *Absorption at E_n.* Having defined $\mathcal{X}^{s,\alpha}_{n,0}(\Omega)$, we now derive a LAP at E_n, $n \in \mathbb{N}^*$, for the restriction of the operator H to $\mathcal{X}^{s,\alpha}_{n,0}(\Omega)$, associated with suitable values of s and α.

Proposition 2.5. *Let K be a compact subset of $\mathbb{C}$ obeying (13) and let $s \in (1/2, +\infty)$. Then, for every $\lambda \in K \cap \mathbb{R}$ and $\alpha \in [0, \min(1, s-1/2))$, both limits $\lim_{z \to \lambda,\ \pm\Im\lambda > 0} R(z)$ exist in the uniform operator topology in $\mathcal{B}(\mathcal{X}^{s,\alpha}_{n,0}(\Omega), (\mathcal{X}^{s,\alpha}_{n,0}(\Omega))')$. Moreover the resolvent $R^\pm$ extends to a Hölder continuous function in $K \cap \overline{\mathbb{C}^\pm}$ with order α. Namely there exists $C > 0$ such that we have:*

$$\forall z, z' \in K \cap \overline{\mathbb{C}^\pm},\ \forall f \in \mathcal{X}^{s,\alpha}_{n,0}(\Omega),\ \|(R^\pm(z) - R^\pm(z'))f\|_{(\mathcal{X}^{s,\alpha}_{n,0}(\Omega))'} \leq C|z - z'|^\alpha \|f\|_{\mathcal{X}^{s,\alpha}_n(\Omega)}.$$

Proof. It is clear from (13) upon mimicking the proof of Proposition 2.2, that $z \mapsto \sum_{m \neq n} r_m(z)$ extends to an α-Hölder continuous function in $K \cap \overline{\mathbb{C}^\pm}$, denoted by $\sum_{m \neq n} r_m^\pm$, obeying

$$(16) \qquad \left\| \sum_{m \neq n} r_m^\pm \right\|_{C^{0,\alpha}(K \cap \overline{\mathbb{C}^\pm})} \leq c \|f\|_{L^{2,s}(\Omega)} \|g\|_{L^{2,s}(\Omega)},$$

for some constant $c > 0$ that depends only on K, s and α.

We turn now to examining r_n. Taking into account that K is bounded, we pick $\epsilon > 0$ so large that $K \cap I_n \subset J := (E_n, E_n + \epsilon)$ and refer once more to the proof of Proposition 2.2. We get that $z \mapsto r_n(z; I_n \setminus \overline{J})$ extends to a Hölder continuous function $r_n^\pm(\cdot; I_n \setminus \overline{J})$ of exponent α in $K \cap \overline{\mathbb{C}^\pm}$, with

$$(17) \qquad \left\| r_n^\pm(\cdot; I_n \setminus \overline{J}) \right\|_{C^{0,\alpha}(K \cap \overline{\mathbb{C}^\pm})} \leq c' \|f\|_{L^{2,s}(\Omega)} \|g\|_{L^{2,s}(\Omega)},$$

where $c' > 0$ is a constant depending only on K, s, α, and ϵ.

Finally, since f and g are taken in $\mathcal{X}^{s,\alpha}_{n,0}(\Omega)$, then the function $\lambda \mapsto H_n(\lambda)$, defined in (7), is α-Hölder continuous in J. Moreover, we have

$$(18) \quad \|H_n\|_{\mathcal{C}^{0,\alpha}(\overline{J})} \leq \|\Lambda_n f_n\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{J})} \|\Lambda_n g_n\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{J})} \leq \|f_n\|_{\mathcal{K}^\alpha_n(\mathbb{R})} \|g_n\|_{\mathcal{K}^\alpha_n(\mathbb{R})} \leq \|f\|_{\mathcal{X}^\alpha_n(\Omega)} \|g\|_{\mathcal{X}^\alpha_n(\Omega)}.$$

Bearing in mind that $H_n(E_n) = 0$ and $E_n + \epsilon \notin K$, we deduce from (18) and Lemma 2.1 that $z \mapsto r_n^{\pm}(\cdot; J)$ extends to an α-Hölder continuous function, still denoted by $r_n^{\pm}(\cdot; J)$, in $K \cap \overline{\mathbb{C}^{\pm}}$, obeying

$$\|r_n^{\pm}(\cdot; J)\|_{\mathcal{C}^{0,\alpha}(K\cap\overline{\mathbb{C}^{\pm}})} \le c'\|H_n\|_{\mathcal{C}^{0,\alpha}(\overline{J})} \le c'\|f\|_{\mathcal{X}_n^{s,\alpha}(\Omega)}\|g\|_{\mathcal{X}_n^{s,\alpha}(\Omega)}, \tag{19}$$

where c' is the same as in (17). Finally, putting (16)-(17) and (19) together, we end up getting a constant $C > 0$, which is independent of f and g, such that we have

$$\forall z, z' \in K \cap \overline{\mathbb{C}^{\pm}},\ |\langle R^{\pm}(z)f, g\rangle_{L^2(\Omega)} - \langle R^{\pm}(z')f, g\rangle_{L^2(\Omega)}| \le C|z - z'|^{\alpha}\|f\|_{\mathcal{X}_n^{s,\alpha}(\Omega)}\|g\|_{\mathcal{X}_n^{s,\alpha}(\Omega)}.$$

Here we used the basic identity $r_n = r_n(\cdot; I_n \setminus \overline{J}) + r_n(\cdot; J)$ and the continuity of embedding $\mathcal{X}_{n,0}^{s,\alpha}(\Omega) \hookrightarrow L^{2,s}(\Omega)$. This entails the desired result. ☐

For any compact subset $K \subset \mathbb{C}$, the set $\mathfrak{J}_K := \{m \in \mathbb{N}^*,\ E_m \in K\}$ is finite. Then upon substituting $\mathcal{X}_K^{s,\alpha}(\Omega) := \cap_{m\in\mathfrak{J}_K}\mathcal{X}_{m,0}^{s,\alpha}(\Omega)$ for $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ in the proof of Proposition 2.5, we obtain the following result.

Theorem 2.6. *Let $K \subset \mathbb{C}$ be compact, and let s and α be the same as in Proposition 2.5. Then the resolvent $z \mapsto R^{\pm}(z)$, initially defined on $K \cap \mathbb{C}^{\pm}$, extends to an α-Hölder continuous function on $K \cap \overline{\mathbb{C}^{\pm}}$ in the uniform operator topology in $\mathcal{B}(\mathcal{X}_K^{s,\alpha}, (\mathcal{X}_K^{s,\alpha})')$.*

3. Analytic and decay properties in absorption spaces

In this section we investigate the analytic and decay properties of functions in $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ for $n \in \mathbb{N}^*$, $s \in (1/2, +\infty)$ and $\alpha \in [0, \min(1, s - 1/2))$. We preliminarily establish with the aid of the (explicit) asymptotic behavior of $\lambda_n(k)$ as $k \to +\infty$, that their n-th Fourier coefficient decays super-exponentially fast. This has two main consequences for the n-th harmonic of an arbitrary function in $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$. First, its $L^2(\mathbb{R}^+)$-expectation (with respect to x) is analytically extendable to $\overline{\mathbb{C}^-}$. Second, and more surprisingly, the mean value of this n-th harmonic is a super-exponentially decaying function of the distance to the edge $x = 0$.

As already seen in the derivation of Lemma 2.4, any function $f \in L^{2,s}(\Omega)$ with $s \in (1/2, +\infty)$, whose n-th harmonic f_n, $n \in \mathbb{N}^*$, is lying in $\mathcal{C}_0^{\infty}(\mathbb{R})$, belongs to $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ for every $\alpha \in [0, \min(1, s - 1/2))$. Conversely, we shall see that functions in $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ decay super-exponentially fast at infinity. Actually, this decay property arises from the embedding of $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ in the linear space

$$B_{w_n^{\alpha},\infty} := \{f \in L^2(\Omega),\ w_n^{\alpha}\pi_n(f) \in L^{\infty}(\mathbb{R})\},$$

associated with the weight function

$$w_n^{\alpha}(k) := |\lambda_n(k) - E_n|^{-\alpha}|\lambda_n'(k)|^{-1/2},\ k \in \mathbb{R}, \tag{20}$$

that is established in Proposition 3.2 below.

Remark 3.1. Given $N \in \mathbb{N}^*$, the continuous Besov space (see e.g. [18, Section 10.1]) of order $p > 0$, associated with a suitable weight function $\mu : \mathbb{R}^N \mapsto \mathbb{R}$, is defined as $B_{\mu,p} := \{f \in L^2(\mathbb{R}^N),\ \mu\hat{f} \in L^p(\mathbb{R}^N)\}$, where $\hat{f}$ denotes the Fourier transform of f. The linear space $B_{w_n^{\alpha},\infty}$ is quite reminiscent of these Besov spaces, but it turns out that these two types of sets do not coincide here, since the condition $w_n^{\alpha}f_n \in L^{\infty}(\mathbb{R})$ is imposed solely on the n-th Fourier coefficient

f_n of f, and not on the whole Fourier transform (with respect to y) $\mathcal{F}_y f$, of f. Moreover another difference with the analysis carried out in [18, Section 10.1] is that the weight function w_n^α is not a tempered function.

Proposition 3.2. *For all $n \in \mathbb{N}^*$, $s \in (1/2, +\infty)$ and $\alpha \in [0, \min(1, s-1/2))$, the space $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ is continuously embedded in $B_{w_n^\alpha,\infty}$. More precisely, it holds true that*

$$\forall f \in \mathcal{X}_{n,0}^{s,\alpha}(\Omega),\ \forall k \in \mathbb{R},\ |w_n^\alpha(k) f_n(k)| \leq \|f\|_{\mathcal{X}_n^{s,\alpha}(\Omega)}. \tag{21}$$

Proof. For each $f \in \mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ we have $\mu_n \tilde{f}_n \in \mathcal{C}^{0,\alpha}(\overline{I}_n)$ and $(\mu_n \tilde{f}_n)(E_n) = 0$ from the very definition of $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$, where we recall that $\tilde{f}_n = \Lambda_n \pi_n f$. As a consequence we have

$$|(\mu_n \tilde{f}_n)(\lambda)| \leq \|\tilde{f}_n\|_{\mathcal{C}^{0,\alpha}_{\mu_n}(\overline{I}_n)} |\lambda - E_n|^\alpha,$$

for every $\lambda \in [E_n, +\infty)$. Finally, (21) follows readily from this upon performing the change of variable $\lambda = \lambda_n(k)$ and remembering (14). $\square$

For every $n \in \mathbb{N}^*$, let us now recall from [16, Theorem 1.4] the asymptotic behavior as $k \to +\infty$ of the two functions

$$\begin{cases} \lambda_n(k) = E_n + C_n k^{2n-1} e^{-k^2}(1 + O(k^{-2})), \\ \lambda_n'(k) = -2C_n k^{2n} e^{-k^2}(1 + O(k^{-2})), \end{cases}$$

where $C_n := 2^n((n-1)!\sqrt{\pi})^{-1}$. In light of (20), this entails for any $\alpha \in [0,1)$ that

$$w_n^\alpha(k) \underset{k \to +\infty}{=} C_{n,\alpha} k^{-n(2\alpha+1)+\alpha} e^{k^2(\alpha+1/2)}(1 + O(k^{-2})), \tag{22}$$

where $C_{n,\alpha} := 2^{-1/2} C_n^{-\alpha-1/2}$. Therefore, with reference to Proposition 3.2, we deduce from (22) for any $s \in (1/2, +\infty)$ and $\alpha \in [0, \min(1, s-1/2))$, that the n-th Fourier coefficient f_n of $f \in \mathcal{X}_{n,0}^{s,\alpha}(\Omega)$ is a super-exponentially decaying function of the variable k. This has several interesting consequences we shall make precise below.

3.1. **Analyticity.**

It is well known (see e.g. [24]) that suitable decay properties of the Fourier transform $\hat{f}$ of a real analytic function f translate into analytic continuation of f to appropriate subsets of $\mathbb{C}$. Let us now establish that (21)–(22) yield a similar result.

Corollary 3.3. *Let n, s and α be the same as in Proposition 3.2. Then for any $f \in \mathcal{X}_{n,0}^{\alpha}(\Omega)$, the function $y \mapsto \|\Pi_n f(\cdot, y)\|_{L^2(\mathbb{R}_+)}$, initially defined in $\mathbb{R}$, admits an analytic continuation to $\overline{\mathbb{C}^\pm}$.*

Proof. With reference to (2), we have

$$\|\Pi_n f(\cdot, y)\|_{L^2(\mathbb{R}_+)}^2 = \int_{\mathbb{R}^2} e^{iky} f_n(k) \overline{e^{ik'y} f_n(k')} F(k,k') \mathrm{d}k \mathrm{d}k', \tag{23}$$

for any $y \in \mathbb{C}$, where $F(k,k') := \int_{\mathbb{R}_+} u_n(x,k) u_n(x,k') \mathrm{d}x$. Further, taking into account that $|F(k,k')| \leq 1$ for all $(k,k') \in \mathbb{R}^2$ since $x \mapsto u_n(x,k)$ is $L^2(\mathbb{R}_+)$-normalized, we get that

$$|e^{iky} f_n(k) \overline{e^{ik'y} f_n(k')} F(k,k')| \leq e^{-\Im(y)(k+k')} |f_n(k)||f_n(k')|.$$

Now the result follows from this and (21)–(23), upon applying the dominated convergence theorem (see e.g. [24, Section IX.3]). $\square$

Remark 3.4. It is not clear whether the result of Corollary 3.3 remains valid for $y \mapsto \Pi_n f(x,y)$, uniformly in $x \in \mathbb{R}_+^*$. Indeed, this would require that $\|u_n(\cdot,k)\|_{L^\infty(\mathbb{R}_+)}$ be appropriately bounded with respect to $k \in \mathbb{R}$, which does not seem to be the case (it is expected that this quantity behaves like $\lambda_n(k)^{1/4}$ as $k \to -\infty$, as can be seen from the proof of Theorem 3.5, below).

3.2. **Geometric localization.** In this section we investigate the geometric properties of functions in $\mathcal{X}_{n,0}^{s,\alpha}(\Omega)$. Namely we establish that the mean value of their n-th harmonic in $(x,+\infty) \times \mathbb{R}$ decays super-exponentially fast with $x > 0$, provided x is sufficiently large.

Theorem 3.5. *Let n, s and α be the same as in Proposition 3.2. Then for any positive real number $\beta < \min\left(1, \frac{2\alpha+1}{1+\sqrt{2\alpha+1}}\right)$, there exist two constants $C_n(\beta) > 0$ and $L_n(\beta) > 0$, such that we have*

$$\forall f \in \mathcal{X}_{n,0}^{s,\alpha}(\Omega),\ \forall L \geq L_n(\beta), \quad \int_L^{+\infty} \|\Pi_n f(x,\cdot)\|_{L^2(\mathbb{R})}^2 \mathrm{d}x \leq C_n(\beta) \|f\|_{\mathcal{X}_n^{s,\alpha}(\Omega)}^2 e^{-\beta L^2}. \tag{24}$$

The proof of Theorem 3.5 is divided into two parts, presented in Subsections 3.2.1 and 3.2.2. With reference to (2) we have

$$\|\Pi_n f(x,\cdot)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}k, \tag{25}$$

for all $x \in \mathbb{R}_+^*$, by Plancherel's theorem, and we shall examine $\int_{\mathbb{R}_\pm} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}k$ separately. Indeed, the claim of Theorem 3.5 being reminiscent of the super-exponential decay exhibited by the eigenfunctions of Sturm-Liouville operators with a non vanishing potential, such as $\mathfrak{h}(k)$ for $k \in \mathbb{R}_-$ (see e.g. [21]), it will come as no surprise that the decay property of $\int_{\mathbb{R}_-} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}k$ is obtained from the one of $u_n(\cdot,k)$ for $k \in \mathbb{R}_-$. This is the content of Lemma 3.6 in Subsection 3.2.1. However, such a property being no longer valid for $k \in \mathbb{R}_+$, we treat the case of $\int_{\mathbb{R}_+} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}k$ in Subsection 3.2.2, with the aid of Proposition 3.2.

3.2.1. *Super-exponential decay of the eigenfunctions and consequences.*

Lemma 3.6. *Let $f \in L^2(\Omega)$ and $n \geq 1$. Then for any $\beta \in (0,1)$, there exist two constants $c_n(\beta) > 0$ and $\ell_n(\beta) > 0$, depending only on n and β, such that the estimate*

$$\int_{(L,+\infty)\times\mathbb{R}_-} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}x\mathrm{d}k \leq c_n(\beta) \|f\|_{L^2(\Omega)}^2 e^{-\beta L^2},$$

holds uniformly for $L \in [\ell_n(\beta), +\infty)$.

Proof. We fix $k \in \mathbb{R}_-$ and combine the Feynman-Hellmann formula with the Cauchy-Schwarz inequality, getting

$$-\lambda_n'(k) = 2\int_{\mathbb{R}_+} (x-k) u_n(x,k)^2 \mathrm{d}x \leq 2\|(\cdot - k) u_n(\cdot,k)\|_{L^2(\mathbb{R}_+)},$$

since $\|u_n(\cdot,k)\|_{L^2(\mathbb{R}_+)} = 1$. Thus, recalling from the energy estimate

$$\|u_n'(\cdot,k)\|_{L^2(\mathbb{R}_+)}^2 + \|(\cdot - k) u_n(\cdot,k)\|_{L^2(\mathbb{R}_+)}^2 = \lambda_n(k), \tag{26}$$

arising from the eigenvalue equation

$$u_n''(x,k) = q_n(x,k)u_n(x,k), \text{ where } q_n(x,k) := (x-k)^2 - \lambda_n(k), \tag{27}$$

that $\|(\cdot - k)u_n(\cdot,k)\|_{L^2(\mathbb{R}_+)} \leq \lambda_n(k)^{1/2}$, we find out that $-\lambda_n'(k) \leq 2\lambda_n(k)^{1/2}$. An integration over $(k,0)$ then yields $\lambda_n(k)^{1/2} \leq \lambda_n(0)^{1/2} - k$, and hence

$$\forall k \in \mathbb{R}_-,\ \lambda_n(k) \leq (x_n - k)^2, \text{ where } x_n := \lambda_n(0)^{1/2} = (4n-1)^{1/2}, \tag{28}$$

according to [8, Lemma 2.1 (i)]. Further, taking into account that $u_n(0,k) = 0$, we derive from (26) and the $L^2(\mathbb{R}_+)$-normalization of $u_n(\cdot,k)$ that

$$\forall x \in \mathbb{R}_+,\ u_n(x,k)^2 = 2\int_0^x u_n(t,k)u_n'(t,k)dt \leq 2\|u_n'(\cdot,k)\|_{L^2(\mathbb{R}_+)} \leq 2\lambda_n(k)^{1/2}.$$

It follows from this and (28), that

$$\forall k \in \mathbb{R}_-,\ \|u_n(\cdot,k)\|_{L^\infty(\mathbb{R}_+)} \leq 2^{1/2}\lambda_n(k)^{1/4} \leq 2^{1/2}(x_n - k)^{1/2}. \tag{29}$$

The next step involves noticing from (28) that the effective potential $q_n(x,k)$ is non negative for every $x \in [x_n, +\infty)$, uniformly in $k \in \mathbb{R}_-$:

$$\forall x \geq x_n,\ q_n(x,k) \geq (x-k)^2 - (x_n - k)^2 \geq 0. \tag{30}$$

Therefore, the $H^1(\mathbb{R}_+)$-solution $u_n(\cdot,k)$ of the equation (27) satisfies

$$\lim_{x \to +\infty} q_n(x,k)u_n(x,k)^2 = 0, \tag{31}$$

and

$$\forall x > x_n,\ u_n(x,k)u_n'(x,k) < 0, \tag{32}$$

in virtue of [17, Proposition B.1]. Next, by multiplying (27) by $u_n'(x,k)$, integrating over (t,v) for $t > v \geq x_n$, and sending v to $+\infty$, we deduce from (31) that $u_n'(t,k)^2 \leq q_n(t,k)u_n(t,k)^2$ for every $t > x_n$. In view of (32), this entails that

$$\forall x \geq x_n,\ |u_n(x,k)| \leq |u_n(x_n,k)| e^{-\int_{x_n}^x q_n(t,k)^{1/2}\mathrm{d}t}. \tag{33}$$

In light of (30), it holds true that

$$\forall x \geq x_n,\ q_n(x,k)^{1/2} \geq \beta(x - x_n) + (1-\beta^2)^{1/2}(x-x_n)^{1/2}(x_n - k)^{1/2},$$

as we have $a^2 + b^2 \geq (\beta a + (1-\beta^2)^{1/2} b)^2$ for any real numbers a and b. Putting this together with (29) and (33), we get that

$$\forall x \geq x_n,\ \forall k \in \mathbb{R}_-,\ |u_n(x,k)| \leq 2^{1/2}(x_n - k)^{1/2} e^{\frac{2}{3}(1-\beta^2)^{1/2}(x_n-k)^{1/2}(x-x_n)^{3/2}} e^{-\frac{\beta}{2}(x-x_n)^2}.$$

From this and the estimates $L - x_n \geq L/2 \geq x_n$, we infer that

$$\int_L^{+\infty} u_n(x,k)^2 \mathrm{d}x \leq 2(x_n - k) e^{-\frac{4}{3}(1-\beta^2)^{1/2} x_n^{3/2}(x_n-k)^{1/2}} \int_L^{+\infty} e^{-\beta(x-x_n)^2}\mathrm{d}x.$$

Now, taking into account that $2\beta^{1/2}\int_L^{+\infty} e^{-\beta(x-x_n)^2}\mathrm{d}x \leq \pi^{1/2} e^{-\beta(L-x_n)^2}$, we have

$$\forall L \geq 2x_n,\ \forall k \in \mathbb{R},\ \int_L^{+\infty} u_n(x,k)^2 \mathrm{d}x \leq c_n(\beta) e^{-\beta(L-x_n)^2},$$

where the constant $c_n(\beta) := \sup_{k\leq 0} \left(\frac{\pi}{\beta}\right)^{1/2} (x_n - k) e^{-\frac{4}{3}(1-\beta^2)^{1/2} x_n^{3/2}(x_n-k)^{1/2}} < \infty$. Therefore, upon eventually shortening β, we may choose $\ell_n(\beta) \geq 2x_n$ so large that the estimate

$$\int_{(L,+\infty)\times\mathbb{R}_-} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}x\mathrm{d}k \leq c_n(\beta) e^{-\beta L^2} \int_{-\infty}^0 |f(k)|^2 \mathrm{d}k \leq c_n(\beta) \|f\|^2_{L^2(\Omega)} e^{-\beta L^2},$$

holds for every $L \geq \ell_n(\beta)$. This terminates the proof. □

The method used in the derivation of Lemma 3.6 does not work for $\int_{\mathbb{R}_+} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}k$, as the effective potential $q_n(\cdot,k)$ of $\mathfrak{h}(k)$, for $k \in \mathbb{R}_+$, vanishes in $\mathbb{R}_+^*$. Nevertheless we shall see in this case that the geometric localization of $x \mapsto \int_{\mathbb{R}_+} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}k$ relies on the decay properties (21)–(22) of the Fourier coefficient f_n, arising from the assumption $f \in \mathcal{X}_{n,0}^{s,\alpha}(\Omega)$.

3.2.2. *Completion of the proof of Theorem 3.5.* For $L > 0$ fixed, we introduce $k(L) > 0$ (we shall make precise further), such that $k(L) \to +\infty$ as $L \to +\infty$, and notice from (25) that

$$\int_L^{+\infty} \|\Pi_n(x,\cdot)\|^2_{L^2(\mathbb{R})} \mathrm{d}x = \sum_{j=0}^{2} \mathcal{I}_j, \tag{34}$$

where

$$\mathcal{I}_j := \int_{(L,+\infty)\times I_j} u_n(x,k)^2 |f_n(k)|^2 \mathrm{d}x\mathrm{d}k,\ j \in \{0,1,2\},$$

with $I_0 := (-\infty, 0)$, $I_1 := (0, k(L))$, and $I_2 := (k(L), +\infty)$. The term $\mathcal{I}_0$ is already estimated in Lemma 3.6. We shall treat each $\mathcal{I}_j$, $j = 1, 2$, separately. We start with $j = 2$. Actually, it is clear from (21)-(22) that we have

$$\mathcal{I}_2 \leq \int_{k(L)}^{+\infty} |f_n(k)|^2 \mathrm{d}k \leq C \|f\|^2_{\mathcal{X}_n^{\alpha,s}(\Omega)} \int_{k(L)}^{+\infty} k^{2(2\alpha+1)n} e^{-(2\alpha+1)k^2} \mathrm{d}k,$$

for L sufficiently large, so we get

$$\mathcal{I}_2 \leq C \|f\|_{\mathcal{X}_n^{\alpha,s}(\Omega)} k(L)^{2(2\alpha+1)n-1} e^{-(2\alpha+1)k(L)^2}, \tag{35}$$

by standard computations. Here and henceforth, C denotes some generic positive constant that is independent of L and f.

Next, bearing in mind that the Agmon distance associated with the operator $\mathfrak{h}(k)$ is $x \mapsto (x-k)^2/2$, and that $\lambda_n(k) \leq 4n - 1$ for all $k \geq 0$, according to [8, Lemma 2.1 (i), (ii)], we obtain that

$$\forall \beta \in (0,1),\ \exists C > 0,\ \forall (x,k) \in \mathbb{R}_+ \times \mathbb{R}_+,\ |u_n(x,k)| \leq C e^{-\beta \frac{(x-k)^2}{2}},$$

from standard Agmon estimates (see e.g. [2, 14]). Further, assuming that $k(L) < L$, we derive from the above estimate that

$$\mathcal{I}_1 \leq C \int_L^{+\infty} e^{-\beta(x-k(L))^2} \mathrm{d}x \int_0^{k(L)} |f_n(k)|^2 \mathrm{d}k \leq \frac{C}{L - k(L)} e^{-\beta(L-k(L))^2} \|f\|^2_{L^2(\Omega)}, \tag{36}$$

for any fixed $\beta \in (0,1)$. Thus, taking

$$k(L) := \gamma L,\ \text{where } \gamma = \gamma(\alpha,\beta) := \frac{\sqrt{\beta}}{1 + \sqrt{2\alpha+1}} \in (0,1),$$

in such a way that the exponential terms appearing in the upper bound of (35)-(36) coincide, we obtain that

$$\sum_{j=1,2} \mathcal{I}_j \leq C L^{2(2\alpha+1)n-1} e^{-\beta \frac{(2\alpha+1)}{1+\sqrt{2\alpha+1}} L^2} \|f\|^2_{\mathcal{X}_n^{\alpha,s}(\Omega)}.$$

Here we used the continuous embedding of $\mathcal{X}_{n,0}^{\alpha,s}(\Omega)$ in $L^2(\Omega)$. This and Lemma 3.6 yield that $\sum_{j=0,1,2} \mathcal{I}_j \leq C e^{-\beta \min\left(1, \frac{(2\alpha+1)}{1+\sqrt{2\alpha+1}}\right) L^2} \|f\|^2_{\mathcal{X}_n^{\alpha,s}(\Omega)}$, provided L is sufficiently large, so the desired result follows from (34).

Remark 3.7. Notice from Subsection 3.2.2 that the second ingredient of the proof of Theorem 3.5 is the decay property of the Fourier coefficient f_n (in the k variable, i.e. the Fourier variable associated with y) which translates into super-exponential decay of $\Pi_n f$ in the x-variable. This is due to a purely magnetic effect arising from the mixing of the space variable x with the frequency k in phase space, through the symbol of the operator H.

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UNIVERSITÉ DE BORDEAUX, IMB, UMR 5251, 33405 TALENCE CEDEX, FRANCE

E-mail address: Nicolas.Popoff@math.u-bordeaux1.fr

AIX MARSEILLE UNIVERSITÉ, CNRS, CPT UMR 7332, 13288 MARSEILLE, FRANCE & UNIVERSITÉ DE TOULON, CNRS, CPT UMR 7332, 83957 LA GARDE, FRANCE

E-mail address: eric.soccorsi@univ-amu.fr