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Global uniqueness in an inverse problem for time fractional diffusion equations

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Abstract

Given (M, g), a compact connected Riemannian manifold of dimension $d \geqslant 2$, with boundary ∂M, we consider an initial boundary value problem for a fractional diffusion equation on $(0, T) \times M$, $T > 0$, with time-fractional Caputo derivative of order $\alpha \in (0, 1) \cup (1, 2)$. We prove uniqueness in the inverse problem of determining the smooth manifold (M, g) (up to an isometry), and various time-independent smooth coefficients appearing in this equation, from measurements of the solutions on a subset of ∂M at fixed time. In the "flat" case where M is a compact subset of $\mathbb{R}^d$, two out the three coefficients ρ (density), a (conductivity) and q (potential) appearing in the equation $\rho \partial_t^\alpha u - \operatorname{div}(a \nabla u) + qu = 0$ on $(0, T) \times M$ are recovered simultaneously.

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1. Introduction

1.1. Statement of the problem

Let (M, g) be a compact connected Riemannian manifold of dimension $d \geqslant 2$, with boundary ∂M. For a strictly positive function μ we consider the weighted Laplace–Beltrami operator

$$\Delta_{g,\mu} := \mu^{-1} \mathrm{div}_g \, \mu \, \nabla_g,$$

where div_g (resp., ∇_g) denotes the divergence (resp., gradient) operator on (M, g), and $\mu^{\pm 1}$ stands for the multiplier by the function $\mu^{\pm 1}$. If μ is identically 1 in M then $\Delta_{g,\mu}$ coincides with the usual Laplace–Beltrami operator on (M, g). In local coordinates, we have

$$\Delta_{g,\mu} u = \sum_{i,j=1}^{d} \mu^{-1} |g|^{-1/2} \partial_{x_i} (\mu |g|^{1/2} g^{ij} \partial_{x_j} u), \quad u \in C^\infty(M),$$

where $g^{-1} := (g^{ij})_{1 \leqslant i,j \leqslant d}$ and $|g| := \det g$. For $\alpha \in (0, 2)$ we consider the initial boundary value problem (IBVP)

$$\begin{cases} \partial_t^\alpha u - \Delta_{g,\mu} u + qu = 0, & \text{in } (0,T) \times M, \\ u = f, & \text{on } (0,T) \times \partial M, \\ \partial_t^k u(0, \cdot) = 0, & \text{in } M, \ k = 0, ..., m, \end{cases} \tag{1.1}$$

with non-homogeneous Dirichlet data f. Here $m := [\alpha]$ denotes the integer part of α and ∂_t^α is the Caputo fractional derivative of order α with respect to t, defined by

$$\partial_t^\alpha u(t,x) := \frac{1}{\Gamma(m+1-\alpha)} \int_0^t (t-s)^{m-\alpha} \partial_s^{m+1} u(s,x) \mathrm{d}s, \ (t,x) \in Q, \tag{1.2}$$

where Γ is the usual Gamma function expressed as $\Gamma(z) := \int_0^{+\infty} e^{-t} t^{z-1} \mathrm{d}t$ for all $z \in \mathbb{C}$ such that $\Re z > 0$. The system (1.1) models anomalous diffusion phenomena. In the sub-diffusive case $\alpha \in (0, 1)$, the first line in (1.1) is usually named fractional diffusion equation, while in the super-diffusive case $\alpha \in (1, 2)$, it is referred as fractional wave equation.

Given two nonempty open subsets S_{in} and S_{out} of ∂M, $T_0 \in (0, T)$ and $\alpha \in (0, 2)$, we introduce the function space

$$\mathcal{H}_{\mathrm{in},\alpha,T_0} := \{ f \in C^{[\alpha]+1}([0,T], H^{\frac{3}{2}}(\partial M)); \ \mathrm{supp}\, f \subset (0, T_0) \times S_{\mathrm{in}} \},$$

where we recall that $[\alpha]$ stands for the integer part of α. As established in Section 2, problem (1.1) associated with $f \in \mathcal{H}_{\mathrm{in},\alpha,T_0}$ is well posed and the partial Dirichlet-to-Neumann (DN) map

$$\Lambda_{M,g,\mu,q} : \mathcal{H}_{\mathrm{in},\alpha,T_0} \ni f \mapsto \partial_\nu u(T_0, \cdot)_{|S_{\mathrm{out}}} := \sum_{i,j=1}^{d} g^{ij} \nu_i \partial_{x_j} u(T_0, \cdot)_{|S_{\mathrm{out}}}, \tag{1.3}$$

where u denotes the solution to (1.1) and ν is the outward unit normal vector field along the boundary ∂M, is linear bounded from $\mathcal{H}_{\mathrm{in},\alpha,T_0}$ into $L^2(S_{\mathrm{out}})$.

In this paper we examine the problem whether knowledge of $\Lambda_{M,g,\mu,q}$ determines the Riemannian manifold (M, g), and the functions μ and q, uniquely.

1.2. *Physical motivations*

Recall that fractional diffusion equations with time fractional derivatives of the form (1.1) describe several physical phenomena related to anomalous diffusion such as diffusion of substances in heterogeneous media, diffusion of fluid flow in inhomogeneous anisotropic porous media, turbulent plasma, diffusion of carriers in amorphous photoconductors, diffusion in a turbulent flow, a percolation model in porous media, fractal media, various biological phenomena and finance problems (see [12]). In particular, it is known (e.g., [1]) that the classical diffusion–advection equation does not often interpret field data of diffusion of substances in the soil, and as one model equation, the fractional diffusion equation is used.

The diffusion equation with time fractional derivative is a corresponding macroscopic model equation to the continuous-time random walk (CTRW in short) and is derived from the CTRW (e.g., [50,55]).

In particular, in the case where we consider fractional diffusion equations describing the diffusion of contaminants in a soil, we cannot a priori know governing parameters in (1.1) such as reaction rate of pollutants. Thus for prediction of contamination, we need to discuss our inverse problem of determining these parameters from measurements of the flux on S_{out} at a fixed time $t = T_0$ associated to Dirichlet inputs at S_{in}.

1.3. *State of the art*

Fractional derivative, ordinary and partial, differential equations have attracted attention over the two last decades. See [39,51,57,61] regarding fractional calculus, and [3,26,49], and the references therein, for studies of partial differential equations with time fractional derivatives. More specifically, the well-posedness of problem (1.1) with time-independent coefficients is examined in [4,25,56], and recently, weak solutions to (1.1) have been defined in [38].

There is a wide mathematical literature for inverse coefficients problems associated with the system (1.1) when $\alpha = 1$ or 2. Without being exhaustive, we refer to [11,13,17,18,20,23,34] for the parabolic case $\alpha = 1$ and to [5–10,21,36,42–44,47,53,58–60] for the hyperbolic case $\alpha = 2$. In contrast to parabolic or hyperbolic inverse coefficient problems, there is only a few mathematical papers dealing with inverse problems associated with (1.1) when $\alpha \in (0, 1) \cup (1, 2)$. In the one-dimensional case, [15] uniquely determines the fractional order α and a time-independent coefficient, by Dirichlet boundary measurements. For $d \geqslant 2$, the fractional order α is recovered in [29] from pointwise measurements of the solution as $t \to 0$ or $t \to \infty$. In [56], the authors prove stable determination of the time-dependent prefactor of the source term. In the particular case where $d = 1$ and $\alpha = 1/2$, using a specifically designed Carleman estimate for (1.1), [16,62] derive a stability estimate of a zero order time-independent coefficient, with respect to partial internal observation of the solution. In [45], time-independent coefficients are uniquely identified by the Dirichlet-to-Neumann map obtained by probing the system with inhomogeneous Dirichlet boundary conditions of the form $\lambda(t)g(x)$, where λ is a fixed real-analytic positive function of the time variable. In [22], the authors proved unique determination of a time-dependent parameter appearing in the source term or in a zero order coefficient, from pointwise measurements of the solution over the whole time interval. We mention also [31,48,52], where several inverse problems for partial differential equations with time-and-space fractional derivatives are discussed, and the more recent work [27], where a formulation of the Calderón problem for the fractional stationary Schrödinger equation is considered.

1.4. Main results

The paper contains two main results. Both of them are uniqueness results for inverse coefficients problems associated with (1.1), but related to two different settings. In the first one, (M, g) is a known compact subset of $\mathbb{R}^d$, while in the second one, (M, g) is an unknown Riemannian manifold to be determined. The first setting is not contained in the second one, however, since in the second case, (M, g) and all the other unknown coefficients are assumed to be smooth, while in the first case the regularity assumptions are relaxed considerably.

We begin by considering the case of a connected bounded domain Ω in $\mathbb{R}^d$, $d \geqslant 2$, with $C^{1,1}$ boundary $\partial\Omega$. Let $\rho \in C(\overline{\Omega})$, $V \in L^\infty(\Omega)$ and $a \in C^1(\overline{\Omega})$ fulfill the condition

$$\rho(x) \geqslant c,\ a(x) \geqslant c,\ V(x) \geqslant 0,\ x \in \Omega, \tag{1.4}$$

for some positive constant c. For $M := \overline{\Omega}$, put

$$g := \rho a^{-1} I_d,\ \mu := \rho^{1-d/2}|a|^{1/2},\ \text{and } q := \rho^{-1} V, \tag{1.5}$$

in the first line of (1.1), where I_d denotes the identity matrix in $\mathbb{R}^{d^2}$. Since (M, g) is a Riemannian manifold with boundary such that $\mu|g|^{1/2} = \rho$, $g^{ij} = 0$ if $i \neq j$, and $g^{ii} = \rho^{-1}a$ for $i, j \in \{1, \dots, d\}$, we have

$$\Delta_{g,\mu} u = \rho^{-1}\text{div}\,(a\nabla u),\ u \in C^\infty(\overline{\Omega}).$$

Therefore, (1.1) can be equivalently rewritten as

$$\begin{cases} \rho\partial_t^\alpha u - \text{div}\,(a\nabla u) + Vu &= 0, & \text{in } Q := (0,T)\times\Omega, \\ u &= f, & \text{on } \Sigma := (0,T)\times\partial\Omega, \\ \partial_t^k u(0,\cdot) &= 0, & \text{in } \Omega,\ k = 0,\dots,m. \end{cases} \tag{1.6}$$

As will appear in Section 3 for any arbitrary $\alpha \in (0,1) \cup (1,2)$ and $T_0 \in (0,T)$, the partial DN map

$$\Lambda_{\rho,a,V}:\ \mathcal{H}_{\text{in},\alpha,T_0} \ni f \mapsto a\partial_\nu u(T_0,\cdot)_{|S_{\text{out}}} := a\nabla u(T_0,\cdot)\cdot\nu_{|S_{\text{out}}} \tag{1.7}$$

where u is the solution to (1.6) and ν is the outward unit normal vector to $\partial\Omega$, is bounded from $\mathcal{H}_{\text{in},\alpha,T_0}$ into $L^2(S_{\text{out}})$. Our first result claims that knowledge of $\Lambda_{\rho,a,V}$ uniquely determines two out of the three coefficients ρ, a, and V, which are referred as, respectively, the density, the conductivity, and the (electric) potential.

Theorem 1.1. *Assume that $S_{\text{in}} \cap S_{\text{out}} \neq \emptyset$ and that $S_{\text{in}} \cup S_{\text{out}} = \partial\Omega$. For $j = 1, 2$, let $\rho_j \in L^\infty(\Omega)$, $a_j \in W^{2,\infty}(\Omega)$, and $V_j \in L^\infty(\Omega)$ satisfy (1.4) with $\rho = \rho_j$, $a = a_j$, $V = V_j$. Moreover, let either of the three following conditions be fulfilled:*

(i) $\rho_1 = \rho_2$ *and*

$$\nabla a_1(x) = \nabla a_2(x), \ x \in \partial\Omega. \tag{1.8}$$

(ii) $a_1 = a_2$ *and*

$$\exists C > 0, \ |\rho_1(x) - \rho_2(x)| \leqslant C \mathrm{dist}(x, \partial\Omega)^2, \ x \in \Omega. \tag{1.9}$$

(iii) $V_1 = V_2$ *and* (1.8)–(1.9) *hold simultaneously true.*

Then, $\Lambda_{\rho_1,a_1,V_1} = \Lambda_{\rho_2,a_2,V_2}$ *yields* $(\rho_1, a_1, V_1) = (\rho_2, a_2, V_2)$.

The second result describes the identifiability properties of the Riemannian manifold (M, g) and the functions $\mu \in \mathcal{C}^\infty(M)$ and $q \in \mathcal{C}^\infty(M)$, appearing in the first line of the IBVP (1.1), that can be inferred from $\Lambda_{M,g,\mu,q}$. It is well known that the DN map is invariant under isometries fixing the boundary. Moreover, gauge equivalent coefficients (μ, q) cannot be distinguished by the DN map either. Here and henceforth, (μ_1, q_1) and (μ_2, q_2) are said gauge equivalent if there exists a strictly positive valued function $\kappa \in C^\infty(M)$ satisfying

$$\kappa(x) = 1 \text{ and } \partial_\nu \kappa(x) = 0, \ x \in \partial M \tag{1.10}$$

such that

$$\mu_2 = \kappa^{-2}\mu_1, \quad q_2 = q_1 - \kappa\Delta_{g,\mu_1}\kappa^{-1}. \tag{1.11}$$

Our second statement is as follows.

Theorem 1.2. *For* $j = 1, 2$, *let* (M_j, g_j) *be two compact and smooth connected Riemannian manifolds of dimension* $d \geqslant 2$ *with the same boundary, and let* $\mu_j \in C^\infty(M_j)$ *and* $q_j \in C^\infty(M_j)$ *satisfy* $\mu_j(x) > 0$ *and* $q_j(x) \geq 0$ *for all* $x \in M_j$. *Let* $S_{\mathrm{in}}, S_{\mathrm{out}} \subset \partial M_1$ *be relatively open and suppose that* $\overline{S_{\mathrm{in}}} \cap \overline{S_{\mathrm{out}}} \neq \emptyset$. *Suppose, moreover, that* $g_1 = g_2$, $\mu_1 = \mu_2 = 1$ *and* $\partial_\nu \mu_1 = \partial_\nu \mu_2 = 0$ *on* ∂M_1. *Then,* $\Lambda_{M_1,g_1,\mu_1,q_1} = \Lambda_{M_2,g_2,\mu_2,q_2}$ *yields that* (M_1, g_1) *and* (M_2, g_2) *are isometric and that* (μ_1, q_1) *and* (μ_2, q_2) *are gauge equivalent.*

1.5. Comments

Notice that the absence of global uniqueness result manifested in Theorems 1.1 (in the sense that only two of the three coefficients ρ, a, and V, are recovered) and 1.2 (where the metric g is determined up to an isometry and (μ, q) are identified modulo gauge transformation) arises from one or several natural obstructions to identification in the system under investigation, each of them being induced by an invariance property satisfied by (1.1).

The first obstruction, which can be found both in Theorems 1.1 and 1.2, is due to the invariance of (1.1) under the group of gauge transformations given by (1.11). Indeed, given a strictly positive function $\kappa \in C^\infty(M)$ satisfying (1.10), we observe for any (μ_1, q_1) and (μ_2, q_2) obeying (1.11), that

$$\Delta_{g,\mu_2}(\kappa w) = \kappa\Delta_{g,\mu_1} w + \delta\kappa w, \ w \in C^\infty(M),$$

where $\delta := \kappa^{-1}\Delta_{g,\mu_1}\kappa - 2\kappa^{-2}(\nabla_g\kappa, \nabla_g\kappa)_g$, and $(\cdot,\cdot)_g$ denotes the inner product on (M, g). In particular, taking $w = \kappa^{-1}$ we get the simpler expression $\delta = -\kappa\Delta_{g,\mu_1}\kappa^{-1}$. Finally, taking $w = u$, where u is the solution to (1.1) associated with $\mu = \mu_1$ and $q = q_1$, we find that

$$(\partial_t^\alpha - \Delta_{g,\mu_2} + q_2)(\kappa u) = \kappa(\partial_t^\alpha - \Delta_{g,\mu_1} + q_1)u = 0.$$

Since our assumptions (1.10) on κ imply that $\partial_\nu(\kappa u) = \partial_\nu u$ and $\kappa u = u$ on $(0, T) \times \partial M$, we find that $\Lambda_{M,g,\mu_1,q_1} = \Lambda_{M,g,\mu_2,q_2}$. This proves that the DN map is invariant under the group of gauge transformations

$$(\mu, q) \mapsto (\kappa^{-2}\mu, q - \kappa\Delta_{g,\mu}\kappa^{-1})$$

parametrized by strictly positive functions $\kappa \in C^\infty(M)$ satisfying (1.10). Notice that the conditions $g_1 = g_2$, $\mu_1 = \mu_2 = 1$ and $\partial_\nu\mu_1 = \partial_\nu\mu_2 = 0$ imposed on ∂M_1 in Theorem 1.2 are analogous to (1.9) in Theorem 1.1. Moreover, the above mentioned invariance property of the system indicates that the result of Theorem 1.1, where two of the three coefficients ρ, a, and V, are simultaneously identified while keeping the third one fixed, is the best one could expect.

The second obstruction arises from the fact that (1.1) is invariant with respect to changes of coordinates. That is, if $\Phi : M \to M$ is a diffeomorphism fixing the boundary ∂M then $\Lambda_{M,g,\mu,q} = \Lambda_{M,\Phi^*g,\mu\circ\Phi,q\circ\Phi}$ where Φ^*g is the pullback of g by Φ. Such invariance properties were already discussed in [40].

We stress out that the same inverse problem formulated with the Riemann–Liouville fractional derivatives instead of the Caputo derivatives, can be treated in a similar way by assigning suitable initial values to the system.

The assumption of Theorem 1.1 (resp., Theorem 1.2) that V (resp., q) is non-negative is to guarantee that the eigenvalues of the Laplace–Beltrami (resp., Laplace) operator perturbed by q (resp., V) are positive, which makes the derivation of the result easier. For instance, the fact used below in the proof of Lemmas 2.3 and 3.3, that $p^\alpha + \lambda_k$ is invertible for all $p \in [0, +\infty)$ and uniformly in $k \geqslant 1$, boils down to the positivity of the eigenvalues $\{\lambda_k;\ k \geqslant 1\}$ of the operator defined by (2.1). It is not hard to see that this assumption can actually be removed at the price of greater unessential technical difficulties.

To our best knowledge, the results of this article are the most precise so far, about the recovery of coefficients appearing in a time fractional diffusion equation from boundary measurements. We prove recovery of a wide class of coefficients from partial boundary measurements that consist of an input on the part S_{in} of the boundary and observation of the flux at the part S_{out} for one fixed time $t = T_0 \in (0, T)$. Our results extend the ones contained in the previous work [45] related to this problem. Another benefit of our approach is its generality, which makes it possible to treat the case of a smooth Riemannian manifold, and the one of a bounded domain with weak regularity assumptions on the coefficients.

Notice that (1.6) associated with $\alpha = 1$ is the usual heat equation, in which case Theorem 1.1 is contained in [13,14]. We point out that the strategy used in [13,14] for the derivation of Theorem 1.1 with $\alpha = 1$, cannot be adapted to the framework of time fractional derivative diffusion equations of order $\alpha \in (0, 1) \cup (1, 2)$. This is due to the facts that a solution to a time fractional derivative equation is not described by a semi-group, and that there is only a limited smoothing property, and no integration by parts formula or Leibniz rule, with respect to the time variable, in this context. As a consequence, the analysis developed in this text is quite different from the one carried out by [13,14].

Notice from Theorem 1.2 that the statement of Theorem 1.1 still holds true for smooth coefficients in a smooth domain, under the weaker assumption $\overline{S_{\text{in}}} \cap \overline{S_{\text{out}}} \neq \emptyset$. Nevertheless, in contrast to Theorem 1.2 where we focus on the recovery of the Riemaniann manifold and the metric, the main interest of Theorem 1.1 lies in the weak regularity assumptions imposed on the unknown coefficients of the inverse problem under consideration. In the same spirit, we point out with Theorem 5.3 below, that the result of Theorem 1.2 remains valid when $\overline{S_{\text{in}}} \cap \overline{S_{\text{out}}} = \emptyset$, in the special case where $\mu = 1$ and $q = 0$, and assuming a Hassell-Tao type inequality [28].

The key idea to our proof is the connection between the DN map associated with (1.1) and the boundary spectral data of the corresponding elliptic Schrödinger operator. This ingredient has already been used by several authors in the context of hyperbolic (see e.g. [33,34,42–44]), parabolic (see e.g. [14,34]), and dynamical Schrödinger (see e.g. [34]) equations. Nevertheless, to our best knowledge, there is no such approach for time fractional diffusion equations, available in the mathematical literature. Once the connection between the DN map and the boundary spectral data is established, we obtain Theorems 1.1 and 1.2 by applying a Borg–Levinson type inverse spectral result (see e.g. [14,19,30,32,33,35,37,54]).

1.6. Outline

The paper is organized as follows. The next three sections are devoted to the study of the inverse problem associated with (1.6) in a bounded domain, while the last section contains the analysis of the inverse problem associated with (1.1) on a Riemannian manifold.

More precisely, we establish a connection between Theorem 1.1 and a Borg–Levinson type inverse spectral result in the first part of Section 2. In the second part, we introduce mathematical tools used in the analysis of the direct problem associated with (1.6), which is carried out in Section 3. Then we define the partial DN map $\Lambda_{\rho,a,V}$ at the end of Section 3, and complete the proof of Theorem 1.1 in Section 4. Finally, Section 5 contains the proofs of Theorem 1.2, and the stronger result stated in Theorem 5.3 in the particular case where $\mu = 1$ and $q = 0$.

2. The settings

In this section, we begin the analysis of the inverse problem associated with (1.6), which is the purpose of Theorem 1.1. We first establish the connection between Theorem 1.1 and a suitable version of the Borg–Levinson inverse spectral theorem.

2.1. Borg–Levinson type inverse spectral problem and Theorem 1.1

Let us start by defining the boundary spectral data of the Borg–Levinson inverse spectral theory. The results we have in mind are formulated with data similar to the one used in [54] but we recall from e.g. [41] that Borg–Levinson problems for systems with variable impedance in the Robin boundary condition can be addressed with knowledge of the eigenvalues only.

Boundary spectral data. Given a positive constant c, we assume that $\rho \in L^\infty(\Omega)$ satisfies $\rho(x) \geqslant c > 0$ for a.e. $x \in \Omega$, so the scalar product

$$\langle u, v \rangle_\rho := \int_\Omega \rho(x) u(x) v(x) \mathrm{d}x, \ u, v \in L^2(\Omega),$$

is equivalent to the usual one in $L^2(\Omega)$. We denote by $L^2_\rho(\Omega)$ the Hilbertian space $L^2(\Omega)$ endowed with $\langle\cdot,\cdot\rangle_\rho$.

Next, for a nonnegative $V \in L^\infty(\Omega)$, and for $a \in C^1(\overline{\Omega})$ fulfilling $a(x) \geqslant c > 0$ for every $x \in \Omega$, we introduce the quadratic form

$$h[u] := \int_\Omega \left(a(x)|\nabla u(x)|^2 + V(x)|u(x)|^2\right) \mathrm{d}x,\ u \in \mathrm{Dom}(h) := H^1_0(\Omega),$$

and consider the operator H generated by h in $L^2_\rho(\Omega)$. Since $\partial\Omega$ is $C^{1,1}$, H is self-adjoint in $L^2_\rho(\Omega)$ and acts on its domain as

$$Hu := \rho^{-1}\left(\mathrm{div}\,(a\nabla u) + Vu\right),\ u \in \mathrm{Dom}(H) := H^1_0(\Omega) \cap H^2(\Omega), \tag{2.1}$$

according to [24, Theorem 2.2.2.3].

By the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^2_\rho(\Omega)$, the spectrum $\sigma(H)$ of the operator H is purely discrete. Let $\{\lambda_n;\ n \in \mathbb{N}\}$, where $\mathbb{N} := \{1, 2, \ldots\}$, be the non-decreasing sequence of the eigenvalues (repeated according to multiplicities) of H. Furthermore, we introduce a family $\{\varphi_n;\ n \in \mathbb{N}\}$ of eigenfuctions of the operator H, which satisfy

$$H\varphi_n = \lambda_n \varphi_n,\ n \in \mathbb{N}, \tag{2.2}$$

and form an orthonormal basis in $L^2_\rho(\Omega)$. Notice that each φ_n is a solution to the following Dirichlet problem:

$$\begin{cases} -\mathrm{div}\,(a\nabla\varphi_n) + V\varphi_n &= \lambda_n\rho\varphi_n, & \text{in } \Omega, \\ \varphi_n &= 0, & \text{on } \partial\Omega, \\ \int_\Omega \rho(x)\,|\varphi_n(x)|^2\,\mathrm{d}x &= 1, & \end{cases} \tag{2.3}$$

Put $\psi_n := (a\partial_\nu\varphi_n)_{|\partial\Omega}$ for every $n \in \mathbb{N}$. Following [13,14,33], we define the boundary spectral data associated with (ρ, a, V), as

$$\mathrm{BSD}(\rho, a, V) := \{(\lambda_n, \psi_n);\ n \geqslant 1\}.$$

A strategy for the proof of Theorem 1.1. We first recall from [14, Corollaries 1.5, 1.6 and 1.7] the following Borg–Levinson type theorem.

Proposition 2.1. *Under the conditions of Theorem 1.1, assume that either of the three assumptions (i), (ii) or (iii) is verified. Then* $\mathrm{BSD}(\rho_1, a_1, q_1) = \mathrm{BSD}(\rho_2, a_2, q_2)$ *entails that* $(\rho_1, a_1, q_1) = (\rho_2, a_2, q_2)$.

In view of the inverse spectral result stated in Proposition 2.1, we may derive the claim of Theorem 1.1 upon showing that two sets of admissible coefficients (ρ_j, a_j, V_j), $j = 1, 2$, have same boundary spectral data, provided their boundary operators Λ_{ρ_j,a_j,V_j} coincide. Otherwise stated, the proof of Theorem 1.1 is a byproduct of Proposition 2.1 combined with the coming result:

Theorem 2.2. *For $j = 1, 2$, let $V_j \in L^\infty(\Omega)$, $\rho_j \in L^\infty(\Omega)$ and $a_j \in \mathcal{C}^1(\overline{\Omega})$ satisfy (1.4) with $\rho = \rho_j$, $a = a_j$, $V = V_j$. Then $\Lambda_{\rho_1,a_1,V_1} = \Lambda_{\rho_2,a_2,V_2}$ implies $BSD(\rho_1, a_1, V_1) = BSD(\rho_2, a_2, V_2)$, up to an appropriate choice of the eigenfunctions of the operator H_1 defined in (2.1) and associated with $(\rho, a, V) = (\rho_1, a_1, V_1)$.*

Therefore, we are left with the task of proving Theorem 2.2.

2.2. Technical tools

Fractional powers of H. Since H is a strictly positive operator, for all $s \geqslant 0$, we can define H^s by

$$H^s h = \sum_{n=1}^{+\infty} \langle h, \varphi_n\rangle \lambda_n^s \varphi_n, \quad h \in D(H^s) = \left\{ h \in L^2(\Omega);\ \sum_{n=1}^{+\infty} |\langle h, \varphi_n\rangle|^2 \lambda_n^{2s} < \infty \right\}$$

and we consider on $D(H^s)$ the norm

$$\|h\|_{D(H^s)} = \left(\sum_{n=1}^{+\infty} |\langle h, \varphi_n\rangle|^2 \lambda_n^{2s} \right)^{\frac{1}{2}}, \quad h \in D(H^s).$$

Two parameters Mittag–Leffler function. Let α and β be two positive real numbers. Following [61, Section 1.2.1, Eq. (1.56)], we define the Mittag–Leffler function associated with α and β, by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \ z \in \mathbb{C}. \tag{2.4}$$

In the particular framework of this paper, where $\alpha \in (0, 2)$, we recall for further reference from [61, Theorem 1.4] the three following useful estimates.

The first estimate, which holds for any $\alpha \in (0, 2)$ and $\beta \in (0, +\infty)$, claims that there exists a constant $c > 0$, depending only on α and β, such that we have

$$|E_{\alpha,\beta}(-t)| \leqslant \frac{c}{1+t}, \ t \in (0, +\infty). \tag{2.5}$$

The second estimate applies for $\alpha = \beta \in (0, 2)$ and states for every $\theta \in (\pi\alpha/2, \pi\alpha)$, that

$$|E_{\alpha,\alpha}(z)| \leqslant \frac{c}{1+|z|^2}, \tag{2.6}$$

whenever $z \in \mathbb{C} \setminus \{0\}$ satisfies $|\arg(z)| \in [\theta, \pi]$. Here c is a positive constant depending only on α and θ. In contrast to (2.5), which is explicitly stated at formula (1.148) of [61, Theorem 1.6], estimate (2.6) follows from the asymptotic behavior of $E_{\alpha,\alpha}(z)$ as $|z| \to +\infty$ given by formula (1.143) of [61, Theorem 1.4]. Indeed, this formula implies that for $|\arg(z)| \in [\theta, \pi]$ we have

$$E_{\alpha,\beta}(z) = -\frac{z^{-1}}{\Gamma(\alpha-\beta)} + \underset{|z|\to+\infty}{\mathcal{O}}\left(|z|^{-2}\right)$$

and using the fact that $\frac{z^{-1}}{\Gamma(\alpha-\beta)} = 0$ for $\alpha = \beta$ we deduce (2.6).

Finally, the third estimate we shall need in the derivation of Theorem 1.1, follows readily from (2.6) and reads:

$$|E_{\alpha,\alpha}(-t)| \leqslant \frac{c}{1+t^2},\ t \in (0,+\infty). \tag{2.7}$$

An *a priori* elliptic estimate. In what follows, we shall make use several times of the following result.

Lemma 2.3. *Let $\rho \in L^\infty(\Omega)$, $a \in C^1(\overline{\Omega})$ and $V \in L^\infty(\Omega)$ fulfill (1.4). Then there exists a constant $c > 0$, such that the estimate*

$$\sum_{n=1}^{+\infty} \lambda_n^{-2} \left| \int_{\partial\Omega} g(x)\psi_n(x)dx \right|^2 \leqslant c\|g\|^2_{H^{3/2}(\partial\Omega)}, \tag{2.8}$$

holds whenever $g \in H^{3/2}(\partial\Omega)$.

Proof. We first prove that the boundary value problem

$$\begin{cases} -\mathrm{div}\,(a\nabla v) + Vv &= 0, \quad \text{in } \Omega, \\ v &= g, \quad \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

admits a unique solution $v \in H^2(\Omega)$. To do that, we refer to [46, Section 1, Theorem 8.3] and pick $G \in H^2(\Omega)$ satisfying $G = g$ on $\partial\Omega$ and the estimate

$$\|G\|_{H^2(\Omega)} \leqslant C\|g\|_{H^{3/2}(\partial\Omega)}. \tag{2.10}$$

Here and in the remaining of the proof C denotes a positive constant that does not depend on g. Evidently v is a solution to (2.9) if and only if $w = v - G$ is a solution to

$$\begin{cases} -\mathrm{div}\,(a\nabla w) + Vw &= F_G, \quad \text{in } \Omega, \\ w &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.11}$$

where $F_G := -(-\mathrm{div}\,(a\nabla G) + VG)$. Since H is boundedly invertible in $L^2(\Omega)$ and $\rho^{-1}F_G \in L^2(\Omega)$, $w = H^{-1}(\rho^{-1}F_G) \in \mathrm{Dom}(H) = H_0^1(\Omega) \cap H^2(\Omega)$ is the unique solution to (2.11), and we have

$$\|w\|_{H^2(\Omega)} \leqslant C\|F_G\|_{L^2(\Omega)}. \tag{2.12}$$

Here we used the fact, arising from the strict ellipticity of H (see [24, Sections 2.2, 2.3, and 2.4]), that the graph norm of H is equivalent to the usual norm in $H^2(\Omega)$. Therefore, $v = w + G \in H^2(\Omega)$ is the unique solution to (2.9) and we derive from (2.10) and (2.12) that

$$\|v\|_{H^2(\Omega)} \leqslant C\|g\|_{H^{3/2}(\partial\Omega)}. \tag{2.13}$$

Now, in view of (2.9), we get for each $n \in \mathbb{N}$ that

$$0 = \langle -\mathrm{div}\,(a\nabla v) + Vv, \varphi_n\rangle = \langle v, H\varphi_n\rangle_\rho + \int_{\partial\Omega} g(x)\psi_n(x)\mathrm{d}x,$$

upon integrating by parts twice. This and (2.2) yield that

$$v_n := \langle v, \varphi_n\rangle_\rho = -\lambda_n^{-1}\int_{\partial\Omega} g(x)\psi_n(x)\mathrm{d}x,\ n \in \mathbb{N}. \tag{2.14}$$

Finally, putting the Parseval identity $\sum_{n=1}^{+\infty}|v_n|^2 = \|v\|_\rho^2$ together with (2.13)–(2.14), we obtain (2.8). $\square$

3. Analysis of the direct problem

In this section we rigorously define the DN map (1.7), which requires that the direct problem associated with (1.6) be preliminarily examined. Next we relate the DN map (1.7) to the BSD.

We start with the sub-diffusive case $\alpha \in (0, 1)$.

Proposition 3.1. *Let $\alpha \in (0, 1)$, $\rho \in L^\infty(\Omega)$, $a \in C^1(\overline{\Omega})$ and $V \in L^\infty(\Omega)$ fulfill (1.4), and let $f \in C^1([0, T], H^{\frac{3}{2}}(\partial\Omega))$ satisfy $f(0, \cdot) = 0$ on $\partial\Omega$. Then, there exists a unique solution $u \in C([0, T], L^2(\Omega))$ to the boundary value problem (1.6). Moreover, we have $u \in C((0, T], H^{2\gamma}(\Omega))$ for any $\gamma \in (0, 1)$.*

Proof. With reference to [46, Section 1, Theorem 8.3] we pick $F \in C^1([0, T], H^2(\Omega))$ satisfying $F = f$ on Σ. Then, it is apparent that u is a solution to (1.6) if and only if $v := u - F$ is a solution to the IBVP

$$\begin{cases} \rho\partial_t^\alpha v - \mathrm{div}\,(a\nabla v) + Vv &= G, & \text{in } Q, \\ v &= 0, & \text{on } \Sigma, \\ v(0,\cdot) &= v_0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $G := -(\rho\partial_t^\alpha F - \nabla\cdot a\nabla F + VF)$ and $v_0 := -F(0, \cdot)$. Applying the Laplace transform to (3.1) we find through basic computations similar to the ones used in the derivation of [56, Theorems 2.1 and 2.2], that

$$v(t,\cdot) = S_0(t)v_0 + \int_0^t S(s)G(t-s,\cdot)\mathrm{d}s,\ t \in (0, T), \tag{3.2}$$

where we have set

$$S_0(t)h := \sum_{n=1}^{+\infty} E_{\alpha,1}(-\lambda_n t^\alpha)\langle h, \varphi_n\rangle\varphi_n \text{ and } S(t)h := t^{\alpha-1}\sum_{n=1}^{+\infty} E_{\alpha,\alpha}(-\lambda_n t^\alpha)\langle h, \varphi_n\rangle\varphi_n, \tag{3.3}$$

for every $t \in (0, T)$ and $h \in L^2(\Omega)$. Further, in view of (2.5), we have

$$\|S_0(t)h\|^2_{H^2(\Omega)} \leqslant C \sum_{n=1}^{+\infty} \lambda_n^2 E_{\alpha,1}(-\lambda_n t^\alpha)^2 |\langle h, \varphi_n\rangle|^2$$
$$\leqslant C \sum_{n=1}^{+\infty} t^{-2\alpha} |\langle h, \varphi_n\rangle|^2 = Ct^{-2\alpha} \|h\|^2_{L^2(\Omega)}, \tag{3.4}$$

where C is a positive constant that is independent of t and h. Since the convergence of the series appearing on the right hand side of (3.4) is uniform with respect to $t \in [\varepsilon, T]$, for any fixed $\varepsilon \in (0, T)$, we see that $t \mapsto S_0(t)h \in C([\varepsilon, T], H^2(\Omega))$. And since ε is arbitrary in $(0, T)$, we obtain

$$t \mapsto S_0(t)h \in C((0, T], H^2(\Omega)). \tag{3.5}$$

Similarly, we obtain for all $t \in (0, T)$ and $h \in L^2(\Omega)$ that

$$\|S(t)h\|^2_{H^{2\gamma}(\Omega)} \leqslant Ct^{2(\alpha-1)} \sum_{n=1}^{+\infty} \lambda_n^{2\gamma} E_{\alpha,\alpha}(-\lambda_n t^\alpha)^2 |\langle h, \varphi_n\rangle|^2$$
$$\leqslant C \sum_{n=1}^{+\infty} t^{-2(1-\alpha(1-\gamma))} |\langle h, \varphi_n\rangle|^2 = Ct^{-2(1-\alpha(1-\gamma))} \|h\|^2_{L^2(\Omega)}. \tag{3.6}$$

As a consequence we have $t \mapsto \int_0^t S(s)G(t-s, \cdot)\mathrm{d}s \in C([0, T], H^{2\gamma}(\Omega))$, since $G \in C([0, T], L^2(\Omega))$. This, (3.2) and (3.5) yield that v, and hence u, is lying in $C((0, T], H^{2\gamma}(\Omega))$.

We turn now to proving that $\lim_{t\to 0} \|u(t)\|_{L^2(\Omega)} = 0$, or equivalently that $\lim_{t\to 0} \|v(t) - v_0\|_{L^2(\Omega)} = 0$. With reference to (3.2), we shall actually establish that

$$\lim_{t\to 0} \|S_0(t)v_0 - v_0\|_{L^2(\Omega)} = \lim_{t\to 0} \left\| \int_0^t S(s)G(t-s)\mathrm{d}s \right\|_{L^2(\Omega)} = 0. \tag{3.7}$$

This can be achieved upon recalling from (3.3) that

$$\|S_0(t)v_0 - v_0\|^2_{L^2(\Omega)} = \sum_{n=1} \left(E_{\alpha,1}(-\lambda_n t^\alpha) - 1\right)^2 |\langle v_0, \varphi_n\rangle|^2, \ t \in (0, T), \tag{3.8}$$

noticing that $\lim_{t\to 0} \left(E_{\alpha,1}(-\lambda_n t^\alpha) - 1\right) = 0$ for every $n \in \mathbb{N}$, and taking advantage of the fact that the series on the right hand side of (3.8) converges uniformly with respect to $t \in (0, T)$, as we have

$$|E_{\alpha,1}(-\lambda_n t^\alpha) - 1| \leqslant \frac{c}{1 + \lambda_n t^\alpha} + 1 \leqslant c + 1, \ t \in (0, T), \ n \in \mathbb{N},$$

by (2.5). Further, since

$$\begin{aligned}\left\|\int_0^t S(s)G(t-s)\mathrm{d}s\right\|_{L^2(\Omega)} &\leqslant \int_0^t \|S(t)G(t-s)\|_{H^{2\gamma}(\Omega)}\mathrm{d}s \\ &\leqslant C\int_0^t s^{-1+\alpha(1-\gamma)}\|G(t-s,\cdot)\|_{L^2(\Omega)}\mathrm{d}s \\ &\leqslant (C/\alpha(1-\gamma))t^{\alpha(1-\gamma)}\|G\|_{C([0,T],L^2(\Omega))},\ t\in(0,T),\end{aligned}$$

by (3.6), we end up getting (3.7). This terminates the proof since v, and hence u, is uniquely defined by (3.2). □

In view of Proposition 3.1, for $\alpha \in (0,1)$ and for any $f \in C^1([0,T], H^{\frac{3}{2}}(\partial\Omega))$ such that $f(0,\cdot)=0$ on $\partial\Omega$ and all $\gamma \in (0,1)$, there exists a unique solution $u \in C([0,T], L^2(\Omega)) \cap C((0,T], H^{2\gamma}(\Omega))$ to (1.6). Thus, taking $\gamma \in (3/4,1)$, we see that the mapping

$$a\partial_\nu u :\ [0,T]\times\partial\Omega \ni (t,x) \mapsto a(x)\partial_\nu u(t,x) := a(x)\nabla u(t,x)\cdot \nu(x),$$

where ν denotes the outward unit normal vector to $\partial\Omega$, is well defined in $C((0,T], L^2(\partial\Omega))$. From this result we deduce for all $\alpha \in (0,1)$ that the operator $\Lambda_{\rho,a,V}$ is bounded from $\mathcal{H}_{\mathrm{in},\alpha,T_0}$ into $L^2(S_{\mathrm{out}})$.

Further, arguing as above, we derive the following result in the super-diffusive case $\alpha \in (1,2)$.

Proposition 3.2. *Let $\alpha \in (1,2)$, $\rho \in L^\infty(\Omega)$, $a \in C^1(\overline{\Omega})$ and $q \in L^\infty(\Omega)$ fulfill (1.4), and let $f \in C^2([0,T], H^{\frac{3}{2}}(\partial\Omega))$ satisfy $f(0,\cdot)=\partial_t f(0,\cdot)=0$ on $\partial\Omega$. Then, there exists a unique solution $u \in C([0,T], L^2(\Omega))$ to the boundary value problem (1.6). Moreover, we have $u \in C((0,T], H^{2\gamma}(\Omega))$ for any $\gamma \in (0,1)$.*

Fix $\alpha \in (1,2)$. We deduce from Proposition 3.2 that for all $\gamma \in (3/4,1)$ and all $f \in C^2([0,T], H^{\frac{3}{2}}(\partial\Omega))$ verifying $f(0,\cdot)=\partial_t f(0,\cdot)=0$ on $\partial\Omega$, there exists a unique solution $u \in C([0,T], L^2(\Omega)) \cap C((0,T], H^{2\gamma}(\Omega))$ to (1.6). Therefore, the mapping

$$a\partial_\nu u :\ [0,T]\times\partial\Omega \ni (t,x) \mapsto a(x)\partial_\nu u(t,x),$$

is well defined in $C((0,T], L^2(\partial\Omega))$, and the operator $\Lambda_{\rho,a,V}$ is bounded from $\mathcal{H}_{\mathrm{in},\alpha,T_0}$ into $L^2(S_{\mathrm{out}})$.

3.1. Normal derivative representation formula

In view of deriving the representation formula of $\Lambda_{\rho,a,V}$ given in Proposition 3.4, we start by establishing the following technical result..

Lemma 3.3. *Let $\alpha \in (0,1)\cup(1,2)$. For ρ, a and q as in Proposition 3.1 and $f \in \mathcal{H}_{\mathrm{in},\alpha,T_0}$, the solution u to (1.6) reads*

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x), \tag{3.9}$$

for almost every $x \in \Omega$ and each $t \in [0, T]$, where $u_n(t) := \langle u(t, \cdot), \varphi_n\rangle_\rho$ is expressed as

$$u_n(t) = -\int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha) \left(\int_{\partial\Omega} f(t-s, x)\psi_n(x) d\sigma(x) \right) ds. \tag{3.10}$$

Note that the convergence of (3.9) is considered in the sense of $L^2(\Omega)$ for each $t \in [0, T]$.

Proof. The identity (3.9) follows readily from the fact that u is lying in $C([0, T], L^2(\Omega))$ and that $\{\varphi_n;\ n \in \mathbb{N}\}$ is an orthonormal basis of $L^2_\rho(\Omega)$. Next, upon extending f by zero outside $[0, T] \times \partial\Omega$, i.e. putting $f(t, x) := 0$ for $(t, x) \in (T, +\infty) \times \partial\Omega$, and denoting by u the solution to (1.6) in $(0, +\infty) \times \Omega$, we compute for all $n \in \mathbb{N}$, the Laplace transform $\mathcal{L}[u_n](p)$ of u_n which is well defined for $p \in (0, +\infty)$ according to estimate (2.6) and Propositions 3.1 and 3.2. We get

$$\begin{aligned} \mathcal{L}[u_n](p) &= \int_0^{+\infty} u_n(t)e^{-pt}\mathrm{d}t = \int_\Omega \rho(x) \left(\int_0^{+\infty} u(t, x)e^{-pt}\mathrm{d}t \right) \varphi_n(x)\mathrm{d}x \\ &= \langle \mathcal{L}[u](p, .), \varphi_n\rangle_\rho, \end{aligned} \tag{3.11}$$

through standard computations. Since $\mathcal{L}[\partial_t^\alpha u](p) = p^\alpha \mathcal{L}[u](p)$, by [61, Eq. (2.140)] and the third line of (1.6), we deduce from (3.11) upon applying the Laplace transform on both sides of the first line in (1.6), that

$$\begin{aligned} p^\alpha \mathcal{L}[u_n](p) &= \langle p^\alpha \mathcal{L}[u](p, .), \varphi_n\rangle_\rho \\ &= -\langle -\mathrm{div}\,(a\nabla\mathcal{L}[u](p)) + q\mathcal{L}[u](p), \varphi_n\rangle,\ \ p \in (0, +\infty). \end{aligned} \tag{3.12}$$

Thus, applying the Green formula on the right hand side of (3.12), we get for each $p \in (0, +\infty)$ that

$$p^\alpha \mathcal{L}[u_n](p) = -\lambda_n \langle \mathcal{L}[u](p), \varphi_n\rangle_\rho - \int_{\partial\Omega} \mathcal{L}[f](p, x)\psi_n(x)\mathrm{d}\sigma(x).$$

As a consequence we have

$$\mathcal{L}[u_n](p) = -(p^\alpha + \lambda_n)^{-1} \mathcal{L}\left[\int_{\partial\Omega} f(\cdot, x)\psi_n(x)\mathrm{d}\sigma(x) \right](p),\ \ p \in (0, +\infty),$$

for every $n \in \mathbb{N}$. This, [61, Eq. (1.80)] and the injectivity of the Laplace transform, yield (3.10). □

We turn now to proving the main result of this section.

Proposition 3.4. *Let $\alpha \in (0, 1) \cup (1, 2)$ and let ρ, a, and V, be as in Proposition 3.1. Pick $f \in \mathcal{H}_{\mathrm{in},\alpha,T_0}$, where $T_0 \in (0, T)$ is fixed, and let u be the solution to (1.6). Then, for a.e. $x \in \partial\Omega$, we have*

$$a(x)\partial_\nu u(T_0,x)=\int_0^{T_0} s^{\alpha-1}\left(\sum_{n=1}^{\infty}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\left(\int_{\partial\Omega} f(T_0-s,y)\psi_n(y)d\sigma(y)\right)\psi_n(x)\right)ds. \tag{3.13}$$

Proof. Let us first establish for each $s\in(0,T_0)$ that the series $\sum_{n=1}^{+\infty}\gamma_n(s)\varphi_n$, where

$$\gamma_n(s):=-s^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n s^\alpha)\left(\int_{\partial\Omega} f(T_0-s,y)\psi_n(x)\mathrm{d}\sigma(x)\right),\ n\in\mathbb{N}, \tag{3.14}$$

converges in $H^2(\Omega)$. Actually, since the domain of the operator H is continuously embedded in $H^2(\Omega)$, it is enough to check that $\sum_{n=1}^{+\infty}\lambda_n\gamma_n(s)\varphi_n$ converges in $L^2_\rho(\Omega)$. This can be achieved with the help of (2.7). Indeed, in view of (3.14) we get through elementary computations that

$$|\gamma_n(s)|\leqslant cs^{-(1+\alpha)}\lambda_n^{-2}\left|\int_{\partial\Omega} f(T_0-s,y)\psi_n(x)\mathrm{d}\sigma(x)\right|,\ s\in(0,T_0),\ n\in\mathbb{N}, \tag{3.15}$$

where c is the same as in (2.7). This entails

$$\begin{aligned}\sum_{n=1}^{+\infty}\lambda_n^2|\gamma_n(s)|^2&\leqslant C_\alpha^2 s^{-2(1+\alpha)}\left(\sum_{n=1}^{+\infty}\lambda_n^{-2}\left|\int_{\partial\Omega} f(T_0-s,y)\psi_n(x)\mathrm{d}\sigma(x)\right|^2\right)\\ &\leqslant c^2 s^{-2(1+\alpha)}\|f(T_0-s,\cdot)\|^2_{H^{3/2}(\partial\Omega)},\ s\in(0,T_0),\end{aligned} \tag{3.16}$$

upon applying Lemma 2.3 with $g=f(T_0-s,\cdot)$, for some constant $c>0$, independent of s and f. As a consequence $\sum_{n=1}^{+\infty}\lambda_n\gamma_n(s)\varphi_n$ converges in $L^2_\rho(\Omega)$ for every $s\in(0,T_0)$, and hence $\sum_{n=1}^{+\infty}\gamma_n(s)\varphi_n$ converges in $H^2(\Omega)$.

Next, since $\operatorname{supp} f\subset(0,T_0)\times S_{\mathrm{in}}$, it is apparent that $s\mapsto s^{-(1+\alpha)}\|f(T_0-s,\cdot)\|_{H^{3/2}(\partial\Omega)}\in L^1(0,T_0)$, and similarly, we see from (3.16) that $s\mapsto\sum_{n=1}^{N}\lambda_n\gamma_n(s)\varphi_n\in L^1(0,T_0,L^2_\rho(\Omega))$ for every $N\in\mathbb{N}$, as we have

$$\left\|\sum_{n=1}^{N}\lambda_n\gamma_n(s)\varphi_n\right\|_\rho=\left(\sum_{n=1}^{N}\lambda_n^2|\gamma_n(s)|^2\right)^{1/2}\leqslant Cs^{-(1+\alpha)}\|f(T_0-s,\cdot)\|_{H^{3/2}(\partial\Omega)},\ s\in(0,T_0),$$

with $C>0$, independent of s and N. Therefore, $s\mapsto\sum_{n=1}^{+\infty}\gamma_n(s)\varphi_n\in L^1(0,T_0,H^2(\Omega))$ by (3.16) and the Lebesgue dominated convergence theorem, and the identity

$$\sum_{n=1}^{+\infty}\left(\int_0^{T_0}\gamma_n(s)\mathrm{d}s\right)\varphi_n=\int_0^{T_0}\left(\sum_{n=1}^{+\infty}\gamma_n(s)\varphi_n\right)\mathrm{d}s, \tag{3.17}$$

holds in $H^2(\Omega)$. Recalling from Lemma 3.3 that $u(T_0,\cdot)=\sum_{n=1}^{+\infty}\left(\int_0^{T_0}\gamma_n(s)\mathrm{d}s\right)\varphi_n$ in $L^2_\rho(\Omega)$, then by uniqueness of the limit, we end up getting from (3.17) that the identity

$$u(T_0,\cdot)=\int_0^{T_0}\left(\sum_{n=1}^{+\infty}\gamma_n(s)\varphi_n\right)\mathrm{d}s, \tag{3.18}$$

holds in $H^2(\Omega)$.

Finally, we obtain (3.13) from the continuity of the trace map $w\in H^2(\Omega)\mapsto \partial_\nu w_{|\partial\Omega}\in L^2(\partial\Omega)$ by mimicking all the steps of the derivation of (3.18). □

4. Proof of Theorem 2.2

The proof is divided into 4 steps.

Step 1: Set up. For $j=1,2$, let H_j be the operator defined by (2.1) with $\rho=\rho_j$, $a=a_j$ and $V=V_j$, and let $\{\lambda_{j,n};\,n\in\mathbb{N}\}$ be the strictly increasing sequence of the eigenvalues of H_j. For each $n\in\mathbb{N}$, we denote by $m_{j,n}\in\mathbb{N}$ the algebraic multiplicity of the eigenvalue $\lambda_{j,n}$ and we introduce a family $\{\varphi_{j,n,p};\ p=1,\ldots,m_{j,n}\}$ of eigenfunctions of H_j, which satisfy

$$H_j\varphi_{j,n,p}=\lambda_{j,n}\varphi_{j,n,p},$$

and form an orthonormal basis in $L^2_{\rho_j}(\Omega)$ of the eigenspace of H_j associated with $\lambda_{j,n}$ (i.e. the linear sub-space of $L^2_{\rho_j}(\Omega)$ spanned by $\{\varphi_{j,n,p},\ p=1,\ldots,m_{j,n}\}$). Further, we put for a.e. $(x,y)\in\partial\Omega$,

$$\Theta_{j,n}(x,y):=\sum_{p=1}^{m_{j,n}}\psi_{j,n,p}(x)\psi_{j,n,p}(y),\ \text{ where } \psi_{j,n,p}:=(a\partial_\nu\varphi_{j,n,p})_{|\partial\Omega}.$$

Then, with reference to (1.7) and Proposition 3.4, it holds true for every $f\in\mathcal{H}_{\mathrm{in},T_0}$ that

$$\Lambda_{\rho_j,a_j,V_j}f=\int_0^{T_0}s^{\alpha-1}\left(\sum_{n=1}^{+\infty}E_{\alpha,\alpha}(-\lambda_{j,n}s^\alpha)\left(\int_{\partial\Omega}f(T_0-s,y)\Theta_{j,n}(\cdot,y)\mathrm{d}\sigma(y)\right)\right)\mathrm{d}s.$$

From this and the assumption $\Lambda_{\rho_1,a_1,V_1}=\Lambda_{\rho_2,a_2,V_2}$ it follows for a.e. $x\in S_{\mathrm{out}}$, that

$$\begin{aligned}\int_0^{T_0}s^{\alpha-1}\Bigg(\sum_{n=1}^{+\infty}\int_{\partial\Omega}\Big(&E_{\alpha,\alpha}(-\lambda_{1,n}s^\alpha)\Theta_{1,n}(x,y)\\&-E_{\alpha,\alpha}(-\lambda_{2,n}s^\alpha)\Theta_{2,n}(x,y)\Big)f(T_0-s,y)\mathrm{d}\sigma(y)\Bigg)\mathrm{d}s=0.\end{aligned} \tag{4.1}$$

In view of the integrand appearing on the left hand side of (4.1), we introduce for every $h\in H^{3/2}(\partial\Omega)$ such that $\operatorname{supp}h\subset S_{\mathrm{in}}$, the following function

$$F_{j,h}(z,x) := \sum_{n=1}^{+\infty} E_{\alpha,\alpha}(-\lambda_{j,n}z) \left(\int_{S_{\mathrm{in}}} \Theta_{j,n}(x,y)h(y)\mathrm{d}\sigma(y) \right), \ z \in \mathbb{C},\ x \in S_{\mathrm{out}}. \tag{4.2}$$

Then, the identity (4.1) being valid for every $f \in \mathcal{H}_{\mathrm{in},T_0}$, we find upon taking $f(t,x) = g(t)h(x)$, for $(t,x) \in (0,T_0) \times S_{\mathrm{in}}$, where g is arbitrary in $C_0^\infty(0,T_0)$ and $h \in H^{3/2}(\partial\Omega)$ is, as above, supported in S_{in}, that

$$F_{1,h}(s^\alpha, x) = F_{2,h}(s^\alpha, x),\ s \in (0,T_0),\ x \in S_{\mathrm{out}}. \tag{4.3}$$

Step 2: Analytic continuation. We start by establishing the following technical result.

Lemma 4.1. *Fix $\theta_0 \in (\pi\alpha/2, \pi\alpha)$ and pick $h \in H^{3/2}(\partial\Omega)$ satisfying* $\operatorname{supp} h \subset S_{\mathrm{in}}$. *Then, both $L^2(S_{\mathrm{out}})$-valued functions $z \mapsto F_{j,h}(z,\cdot)$, $j = 1,2$, defined in (4.2), are holomorphic in the subdomain $\mathcal{D}_{\theta_0} := \{z \in \mathbb{C} \setminus \{0\};\ |arg(z)| < \pi - \theta_0\}$.*

Proof. Let j be either 1 or 2. Bearing in mind that

$$\int_{S_{\mathrm{in}}} \Theta_{j,n}(\cdot,y)h(y)\mathrm{d}\sigma(y) = \sum_{p=1}^{m_{j,n}} \left(\int_{S_{\mathrm{in}}} h(y)\psi_{j,n,p}(y)\mathrm{d}\sigma(y) \right) \psi_{j,n,p},\ n \in \mathbb{N},$$

we see upon arguing as in the derivation of Proposition 3.4, that it is enough to show that the $L^2_\rho(\Omega)$-valued function

$$G_j(z) := \sum_{n=1}^{+\infty} E_{\alpha,\alpha}(-\lambda_{j,n}z) \left(\int_{S_{\mathrm{in}}} h(y)\psi_{j,n}(y)\mathrm{d}\sigma(y) \right) \lambda_{j,n}\varphi_n, \tag{4.4}$$

is holomorphic in $\mathcal{D}_{\theta_0}$.

Further, as $\alpha \in (0,2)$ and $|\arg(-z)| \in [\theta_0, \pi]$, we invoke (2.6) and get some positive constant c such that

$$\left| E_{\alpha,\alpha}(-\lambda_{j,n}z) \right| \leqslant \frac{c}{1 + \left|\lambda_{j,n}z\right|^2},\ z \in \mathcal{D}_{\theta_0},\ n \in \mathbb{N}.$$

As a consequence we have

$$\lambda_{j,n}|E_{\alpha,\alpha}(-\lambda_{j,n}z)| \left| \int_{S_{\mathrm{in}}} h(y)\psi_{j,n}(y)\mathrm{d}\sigma(y) \right| \leqslant c|z|^{-2}\lambda_{j,n}^{-1} \left| \int_{\partial\Omega} h(y)\psi_{j,n}(y)\mathrm{d}\sigma(y) \right|. \tag{4.5}$$

Let K be a compact subset of $\mathcal{D}_{\theta_0}$. Due to Lemma 2.3, (4.5) yields that the series appearing on the right hand side of (4.4) converges in $L^2_{\rho_j}(\Omega)$, uniformly in $z \in K$. Next, the mapping $z \mapsto E_{\alpha,\alpha}(-\lambda_{j,n}z) \left(\int_{S_{\mathrm{in}}} h(y)\psi_{j,n}(y)\mathrm{d}\sigma(y) \right) \lambda_{j,n}\varphi_n$ being holomorphic in K for each $n \in \mathbb{N}$, since the

Mittag–Leffler function $E_{\alpha,\alpha}$ is holomorphic in $\mathbb{C}$ from the definition (2.4), we get that G_j is holomorphic in K as well. This entails that G_j is analytic in $\mathcal{D}_{\theta_0}$ since K is arbitrary in $\mathcal{D}_{\theta_0}$. □

Fix $\theta_0 \in (\pi\alpha/2, \pi\alpha)$. Since $F_{1,h}(z,x) = F_{2,h}(z,x)$ for a.e. $x \in S_{\mathrm{out}}$ and all $z \in (0, T_0^\alpha)$, according to (4.3), the same is true for $z \in \mathcal{D}_{\theta_0}$, by Lemma 4.1 in virtue of the unique continuation principle for analytic functions. This yields

$$F_{1,h}(t,x) = F_{2,h}(t,x),\ x \in S_{\mathrm{out}},\ t \in (0,+\infty). \tag{4.6}$$

Having seen this, we turn now to proving that the identity (4.6) yields

$$\lambda_{1,n} = \lambda_{2,n} \text{ and } \Theta_{1,n}(x,y) = \Theta_{2,n}(x,y),\ n \in \mathbb{N},\ (x,y) \in S_{\mathrm{out}} \times S_{\mathrm{in}}. \tag{4.7}$$

This can be done upon computing the Laplace transform of both sides of (4.6).

Step 3: Laplace transform. We fix $N \in \mathbb{N}$, $t \in (0,+\infty)$ and recall that the $L^2(\partial\Omega)$-norm of

$$\begin{aligned}&\sum_{n=1}^{N} E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\left(\int_{S_{\mathrm{in}}} \Theta_{j,n}(\cdot,y)h(y)\mathrm{d}\sigma(y)\right)\\ &\quad= \sum_{n=1}^{N} E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\left(\sum_{p=1}^{m_{j,n}}\left(\int_{S_{\mathrm{in}}} h(y)\psi_{j,n,p}(y)\mathrm{d}\sigma(y)\right)\psi_{j,n,p}\right),\end{aligned}$$

is upper bounded (up to some positive multiplicative constant that depends only on Ω) by the $L^2_{\rho_j}(\Omega)$-norm of $\sum_{n=1}^{N} E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\left(\sum_{p=1}^{m_{j,n}}\left(\int_{S_{\mathrm{in}}} h(y)\psi_{j,n,p}(y)\mathrm{d}\sigma(y)\right)\lambda_{j,n}\varphi_{j,n,p}\right)$. Hence we find that

$$\begin{aligned}&\left\|\sum_{n=1}^{N} t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\left(\int_{S_{\mathrm{in}}} \Theta_{j,n}(\cdot,y)h(y)\mathrm{d}\sigma(y)\right)\right\|_{L^2(S_{\mathrm{out}})}\\ &\leqslant Ct^{\alpha+1}\left(\sum_{n=1}^{N}\lambda_{j,n}^2 E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)^2\left(\sum_{p=1}^{m_{j,n}}\left|\int_{\partial\Omega} h(y)\psi_{j,n,p}(y)\mathrm{d}\sigma(y)\right|^2\right)\right)^{1/2}\\ &\leqslant Ct^{1-\alpha}\left(\sum_{n=1}^{N}\lambda_{j,n}^{-2}\left(\sum_{p=1}^{m_{j,n}}\left|\int_{\partial\Omega} h(y)\psi_{j,n,p}(y)\mathrm{d}\sigma(y)\right|^2\right)\right)^{1/2},\end{aligned}$$

according to (2.7), the constant $C > 0$ depending neither on N, nor on t. Therefore, we have

$$\left\|\sum_{n=1}^{N} t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\left(\int_{S_{\mathrm{in}}} \Theta_{j,n}(\cdot,y)h(y)\mathrm{d}\sigma(y)\right)\right\|_{L^2(S_{\mathrm{out}})} \leqslant Ct^{1-\alpha}\|h\|_{H^{3/2}(\partial\Omega)},$$

by Lemma 2.3, and the Lebesgue dominated convergence theorem for $L^2(S_{\rm out})$-valued functions yields

$$\mathcal{L}\left[\sum_{n=1}^{+\infty} t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\left(\int_{S_{\rm in}}\Theta_{j,n}(x,y)h(y)\mathrm{d}\sigma(y)\right)\right](p)$$
$$=\sum_{n=1}^{+\infty}\left(\int_{S_{\rm in}}\Theta_{j,n}(x,y)h(y)\mathrm{d}\sigma(y)\right)\mathcal{L}\left[t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\right](p),\ x\in S_{\rm out},\ p\in(0,+\infty). \tag{4.8}$$

We are thus left with the task of computing the Laplace transform of $t\mapsto t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)$ on $(0,+\infty)$. We find by standard computations that [61, Eq. (1.80)] implies

$$\begin{aligned}\mathcal{L}\left[t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\right](p)&=\frac{\mathrm{d}^2}{\mathrm{d}p^2}\left(\mathcal{L}\left[t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\right](p)\right)\\&=p^{\alpha-2}\left(\frac{2\alpha^2p^\alpha-\alpha(\alpha-1)(p^\alpha+\lambda_{j,n})}{(p^\alpha+\lambda_{j,n})^3}\right),\end{aligned} \tag{4.9}$$

for all $p>\lambda_{j,n}^{1/\alpha}$. Further, since $p\mapsto\mathcal{L}\left[t^{\alpha+1}E_{\alpha,\alpha}(-\lambda_{j,n}t^\alpha)\right](p)$ is an analytic function of $p\in\{z\in\mathbb{C};\ \Re z>0\}$, then (4.9) holds for every $p\in(0,+\infty)$, and we have

$$\begin{aligned}&\textstyle\sum_{n=1}^{+\infty}\frac{(2\alpha^2p^\alpha-\alpha(\alpha-1)(p^\alpha+\lambda_{1,n}))\int_{S_{\rm in}}\Theta_{1,n}(x,y)h(y)\mathrm{d}\sigma(y)}{(p^\alpha+\lambda_{1,n})^3}\\&=\textstyle\sum_{n=1}^{+\infty}\frac{(2\alpha^2p^\alpha-\alpha(\alpha-1)(p^\alpha+\lambda_{2,n}))\int_{S_{\rm in}}\Theta_{2,n}(x,y)h(y)\mathrm{d}\sigma(y)}{(p^\alpha+\lambda_{2,n})^3},\ p\in(0,+\infty),\ x\in S_{\rm out},\end{aligned} \tag{4.10}$$

by (4.2), (4.6) and (4.8).

Step 4: End of the proof. Consider $\mathcal{O}:=\mathbb{C}\setminus\{-\lambda_{j,n};\ j=1,2,\ n\in\mathbb{N}\}$ and note that, since, for $j=1,2$, $(\lambda_{j,n})_{n\geqslant1}$ is a strictly increasing and unbounded sequence, the set $\mathcal{O}$ is connected. Let K be a compact subset of $\mathcal{O}:=\mathbb{C}\setminus\{-\lambda_{j,n};\ j=1,2,\ n\in\mathbb{N}\}$. Arguing as in the derivation of Lemma 4.1, we see that, for $j=1,2$, the series

$$\sum_{n=1}^{+\infty}\left(\frac{2\alpha^2z-\alpha(\alpha-1)(z+\lambda_{j,n})}{(z+\lambda_{j,n})^3}\right)\left(\int_{S_{\rm in}}\Theta_{j,n}(x,y)h(y)\mathrm{d}\sigma(y)\right),$$

converges uniformly with respect to $z\in K$. Thus, since K is arbitrary in $\mathcal{O}$, the function

$$z\mapsto\sum_{n=1}^{+\infty}\left(\frac{2\alpha^2z-\alpha(\alpha-1)(z+\lambda_{j,n})}{(z+\lambda_{j,n})^3}\right)\left(\int_{S_{\rm in}}\Theta_{j,n}(\cdot,y)h(y)\mathrm{d}\sigma(y)\right),$$

is analytic in $\mathcal{O}$ and we deduce from (4.10), that

$$\begin{aligned}&\sum_{n=1}^{+\infty}\left(\frac{2\alpha^2 z-\alpha(\alpha-1)(z+\lambda_{1,n})}{(z+\lambda_{1,n})^3}\right)\left(\int_{S_{\text{in}}}\Theta_{1,n}(x,y)h(y)\mathrm{d}\sigma(y)\right)\\&=\sum_{n=1}^{+\infty}\left(\frac{2\alpha^2 z-\alpha(\alpha-1)(z+\lambda_{2,n})}{(z+\lambda_{2,n})^3}\right)\left(\int_{S_{\text{in}}}\Theta_{2,n}(x,y)h(y)\mathrm{d}\sigma(y)\right),\ z\in\mathcal{O},\ x\in S_{\text{out}},\end{aligned}\tag{4.11}$$

by the unique continuation principle for analytic functions. Now, putting $\lambda_{*1}:=\min_{(j,n)\in\{1,2\}\times\mathbb{N}}\lambda_{j,n}$, multiplying both sides of (4.11) by $(z+\lambda_{*1})^3$, and sending z to $(-\lambda_{*1})$, we obtain that

$$\lambda_{1,1}=\lambda_{*1}=\lambda_{2,1}\text{ and }\int_{S_{\text{in}}}\Theta_{1,1}(x,y)h(y)\mathrm{d}\sigma(y)=\int_{S_{\text{in}}}\Theta_{2,1}(\cdot,y)h(y)\mathrm{d}\sigma(y)\text{ for a.e. }x\in S_{\text{out}}.$$

Similarly, by induction on $n\in\mathbb{N}$, we find that

$$\lambda_{1,n}=\lambda_{2,n}\text{ and }\int_{S_{\text{in}}}\Theta_{1,n}(x,y)h(y)\mathrm{d}\sigma(y)=\int_{S_{\text{in}}}\Theta_{2,n}(\cdot,y)h(y)\mathrm{d}\sigma(y)\text{ for a.e. }x\in S_{\text{out}},$$

for any function $h\in H^{3/2}(\partial\Omega)$ supported in S_{in}, which yields (4.7). Finally, since $S_{\text{in}}\cup S_{\text{out}}=\partial\Omega$ and $S_{\text{in}}\cap S_{\text{out}}\neq\emptyset$, we end up getting that $\text{BSD}(\rho_1,a_1,V_1)=\text{BSD}(\rho_2,a_2,V_2)$, up to an appropriate choice of the functions $\{\varphi_{1,n},\ n\in\mathbb{N}\}$, from (4.7) and the end of the proof of Theorem 1.1 in [13] (see [13, pages 975–976]).

5. Results on Riemannian manifolds

In this section we prove Theorem 1.2. Then, we establish in the particular case where $\mu=1$ and $q=0$, upon assuming a spectral Hassell–Tao type inequality (see (5.5) below) that the result of Theorem 1.2 remains valid when $\overline{S_{\text{in}}}\cap\overline{S_{\text{out}}}=\emptyset$.

Nevertheless, in the first step of the analysis we assume a slightly more restrictive assumption, i.e. that $S_{\text{in}}\cap S_{\text{out}}\neq\emptyset$, than the one required by Theorem 1.2. We consider the weighted measure μdx, where dx is the Riemannian volume measure, to define the space $L^2(M)$, since $\Delta_{g,\mu}$ is symmetric with respect to the resulting inner product. Let us introduce the elliptic operator A acting on $L^2(M)$ with domain $D(A)=H_0^1(M)\cap H^2(M)$ defined by

$$Ah=-\Delta_{g,\mu}h+qh,\quad h\in D(A).\tag{5.1}$$

By a compact resolvent argument we know that the spectrum of A consists of a non-decreasing sequence of eigenvalues $(\lambda_n)_{n\geqslant 1}$ and we can introduce the associated Hilbertian basis of eigenfunctions $(\varphi_n)_{n\geqslant 1}$. We define the boundary spectral data as

$$\text{BSD}(M,g,\mu,q;\Gamma):=\{(\lambda_n,\psi_{n|\Gamma});\ n\geqslant 1\},$$

where $\Gamma\subset\partial M$ is open and $\psi_n=\partial_\nu\varphi_n$. In view of these BSD, it is easy to see that the results of Sections 2, 3 and 4 remain valid in the framework of Theorem 1.2. In particular, we may repeat the proof of Theorem 2.2 in the present context to obtain:

Theorem 5.1. *Let (M_k, g_k), $k = 1, 2$, be two compact and smooth connected Riemannian manifolds of dimension $d \geqslant 2$ with the same boundary. Let $\mu_k, q_k \in C^\infty(M_k)$ satisfy $\mu_k(x) > 0$ and $q_k(x) \geq 0$ for all $x \in M_k$, $k = 1, 2$. Let $S_{\mathrm{in}}, S_{\mathrm{out}} \subset \partial M_1$ be relatively open and suppose that $S_{\mathrm{out}} \cap S_{\mathrm{in}}$ is nonempty. Suppose, moreover, that $g_1 = g_2$, $\mu_1 = \mu_2 = 1$ and $\partial_\nu \mu_1 = \partial_\nu \mu_2 = 0$ on ∂M_1. Then, the condition $\Lambda_{M_1,g_1,\mu_1,q_1} = \Lambda_{M_2,g_2,\mu_2,q_2}$ implies that, up to an appropriate choice of the eigenfunctions of the operator A_1, we have*

$$\mathrm{BSD}(M_1, g_1, \mu_1, q_1; S_{\mathrm{out}} \cup S_{\mathrm{in}}) = \mathrm{BSD}(M_2, g_2, \mu_2, q_2; S_{\mathrm{out}} \cup S_{\mathrm{in}}). \tag{5.2}$$

It is well-known that (5.2) implies that (M_k, g_k), $k = 1, 2$, are isometric, and that (μ_k, q_k), $k = 1, 2$, are on the same orbit of the group of gauge transformations, and we refer to [33] for a detailed proof. To our knowledge, all the proofs of this result are based on the Boundary Control method. The Boundary Control method was introduced by Belishev in [5] where he solved the inverse boundary value problem for the isotropic wave equation, that is, the equation (1.6) with $\alpha = 2$, $a = 1$ and $q = 0$. The method was generalized to geometric context by Belishev and Kurylev [6], and the inverse boundary spectral problem with partial data as in (5.2) was solved by Katchalov and Kurylev [32]. We mention that a reduction similar to Theorem 5.1 was shown in [34] in the case of the heat equation, that is, the equation (1.1) with $\alpha = 1$.

Let us now consider two further generalizations where the assumption that $S_{\mathrm{out}} \cap S_{\mathrm{in}}$ is nonempty is weakened. These generalizations are based on the observation that the proof of Theorem 2.2 gives:

Theorem 5.2. *Let (M_k, g_k), $k = 1, 2$, be two compact and smooth connected Riemannian manifolds of dimension $d \geqslant 2$ with the same boundary. Let $\mu_k, q_k \in C^\infty(M_k)$ satisfy $\mu_k(x) > 0$ and $q_k(x) \geq 0$ for all $x \in M_k$, $k = 1, 2$. Let $S_{\mathrm{in}}, S_{\mathrm{out}} \subset \partial M_1$ be relatively open, and suppose that $g_1 = g_2$, $\mu_1 = \mu_2 = 1$ and $\partial_\nu \mu_1 = \partial_\nu \mu_2 = 0$ on ∂M_1. Then, the condition $\Lambda_{M_1,g_1,\mu_1,q_1} = \Lambda_{M_2,g_2,\mu_2,q_2}$ implies that*

$$\lambda_{1,n} = \lambda_{2,n} \text{ and } \Theta_{1,n}(x, y) = \Theta_{2,n}(x, y),\ n \in \mathbb{N},\ (x, y) \in S_{\mathrm{out}} \times S_{\mathrm{in}}, \tag{5.3}$$

where, as before, $\{\lambda_{k,n}; n \in \mathbb{N}\}$ is the strictly increasing sequence of the Dirichlet eigenvalues of A_k and

$$\Theta_{k,n}(x, y) := \sum_{p=1}^{m_{k,n}} \psi_{k,n,p}(x)\psi_{k,n,p}(y), \text{ where } \psi_{k,n,p} := \sum_{i,j=1}^{d} g_k^{ij} \nu_i \partial_{x_j} \varphi_{k,n,p}.$$

Here the eigenfunctions $\varphi_{k,n,p}$ are chosen so that $\varphi_{k,n,p}$, $p = 1, \dots, m_{k,n}$, form an orthonormal basis of the eigenspace associated with $\lambda_{k,n}$.

If $\overline{S_{\mathrm{in}}} \cap \overline{S_{\mathrm{out}}} \neq \emptyset$, then the equation (5.3) implies that the boundary spectral data

$$\mathrm{BSD}(M_k, g_k, \mu_k, q_k; S_{\mathrm{out}}), \quad k = 1, 2, \tag{5.4}$$

are gauge equivalent in the sense that there is a constant $\kappa > 0$ such that up to an appropriate choice of the eigenfunctions of the operator A_1, we have $\psi_{1,n,p} = \kappa\psi_{2,n,p}$ on S_{out} for all n and p, see [42, Theorem 4]. In [42] the authors considered only operators A of the form (5.1) with $\mu = 1$ identically. We can actually reduce to this case upon taking $\kappa = \sqrt{\mu}$ in (1.11). The gauge equivalence of the boundary spectral data (5.4) implies that (M_k, g_k), $k = 1, 2$, are isometric and that (μ_k, q_k), $k = 1, 2$, are on the same orbit of the group of gauge transformations, as can be seen by combining the proofs of [33, Theorems 4.33 and 3.37]. This proves Theorem 1.2.

Finally, let us consider the case where S_{in} and S_{out} are allowed to be far apart. Following [43] we assume that $\mu_k = 1$ and $q_k = 0$ identically, and that both (M_k, g_k), $k = 1, 2$, satisfy the spectral inequality

$$\lambda_{k,n} \le C \|\psi_{k,n,p}\|^2_{L^2(S_{\text{in}})}, \tag{5.5}$$

where the constant $C > 0$ is independent of n and p. Hassell and Tao [28] showed that all non-trapping Riemannian manifolds (M_k, g_k) satisfy (5.5) when S_{in} is replaced by ∂M_k. Moreover, (5.5) follows from (and is strictly weaker than) the geometric control condition by Bardos, Lebeau and Rauch [2], see [43]. We will now give a reduction to the result in [43]. Let us denote by $L_{M,g}$ the hyperbolic DN map associated to the Riemannian manifold (M, g) and restricted to $S_{\text{in}} \times S_{\text{out}}$, that is,

$$L_{M,g} f = \partial_\nu u|_{(0,\infty)\times S_{\text{out}}}, \quad f \in C_0^\infty((0,\infty) \times S_{\text{in}}),$$

where u is the solution of (1.1) with $\mu = 1$, $q = 0$ identically and $\alpha = 2$. The map L_{M_k, g_k} has the representation

$$L_{M_k,g_k} f(t,x) = \sum_{n\in\mathbb{N}} \int_0^t \int_{S_{\text{in}}} f(s,y) \frac{\sin(\sqrt{\lambda_{k,n}}(t-s))}{\sqrt{\lambda_{k,n}}} \Theta_{k,n}(x,y)\, dy\, ds,$$

where dy is the Riemannian surface measure on ∂M_1, see e.g. [33, Lemma 3.6]. Hence (5.3) implies that $L_{M_1,g_1} = L_{M_2,g_2}$, and therefore (M_k, g_k), $k = 1, 2$, are isometric [43]. We have shown:

Theorem 5.3. *Let (M_k, g_k), $k = 1, 2$, be two compact and smooth connected Riemannian manifolds of dimension $d \geqslant 2$ with the same boundary. Let $S_{\text{in}}, S_{\text{out}} \subset \partial M_1$ be relatively open, and suppose that $g_1 = g_2$ on ∂M_1. Suppose, moreover, that both (M_k, g_k), $k = 1, 2$, satisfy the spectral inequality (5.5). Then, the condition $\Lambda_{M_1,g_1,1,0} = \Lambda_{M_2,g_2,1,0}$ implies that (M_k, g_k), $k = 1, 2$, are isometric.*

We do not know if Theorem 5.3 holds for operators with varying μ and q, see the discussion in [44, pp. 7-8].

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