Edge Currents for Quantum Hall Systems, I. One-Edge, Unbounded Geometries

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Abstract

Devices exhibiting the integer quantum Hall effect can be modeled by one-electron Schrödinger operators describing the planar motion of an electron in a perpendicular, constant magnetic field, and under the influence of an electrostatic potential. The electron motion is confined to unbounded subsets of the plane by confining potential barriers. The edges of the confining potential barrier create edge currents. In this, the first of two papers, we prove explicit lower bounds on the edge currents associated with one-edge, unbounded geometries formed by various confining potentials. This work extends some known results that we review. The edge currents are carried by states with energy localized between any two Landau levels. These one-edge geometries describe the electron confined to certain unbounded regions in the plane obtained by deforming half-plane regions. We prove that the currents are stable under various potential perturbations, provided the perturbations are suitably small relative to the magnetic field strength, including perturbations by random potentials. For these cases of one-edge geometries, the existence of, and the estimates on, the edge currents imply that the corresponding Hamiltonian has intervals of absolutely continuous spectrum. In the second paper of this series, we consider the edge currents associated with two-edge geometries describing bounded, cylinder-like regions, and unbounded, strip-like, regions.

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1 Introduction and Main Results

The integer quantum Hall effect (IQHE) refers to the quantization of the Hall conductivity in integer multiples of $2\pi e^2/h$. The IQHE is observed in planar quantum devices at zero temperature and can be described by a Fermi gas of noninteracting electrons. This simplification reduces the study of the dynamics to the one-electron approximation. Typically, experimental devices consist of finitely-extended, planar samples subject to a constant perpendicular magnetic field B. An applied electric field in the x-direction induces a current in the y-direction, the Hall current, and the Hall conductivity σ_{xy} is observed to be quantized. Furthermore, the Hall conductivity is a function of the electron Fermi energy, or, equivalently, the electron filling factor, and plateaus of the Hall conductivity are observed as the filling factor is increased. It is now accepted that the occurrence of the plateaus is due to the existence of localized states near the Landau levels that are created by the random distribution of impurities in the sample, cf. [1], [2].

Another new phenomenon that arises in the study of these devices exhibiting the IQHE is the occurrence of *edge currents* associated with the boundaries of quantum devices. These edge currents are the subject of this work. In order to explain their origin, we recall the theory of an electron in $\mathbb{R}^2$ subject to a constant, transverse magnetic field. The Landau Hamiltonian $H_L(B)$ describes a charged particle constrained to $\mathbb{R}^2$, and moving in a constant, transverse magnetic field with strength $B \geq 0$. Let $p_x = -i\partial_x$ and $p_y = -i\partial_y$ be the two momentum operators. The operator $H_L(B)$ is defined on the dense domain $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ by

$$H_L(B) = (-i\nabla - A)^2 = p_x^2 + (p_y - Bx)^2, \tag{1.1}$$

in the Landau gauge for which the vector potential is $A(x, y) = (0, Bx)$. The map (1.1) extends to a self-adjoint operator with point spectrum given by $\{E_n(B) = (2n + 1)B \mid n = 0, 1, 2, \ldots\}$, called the *Landau levels*, and each eigenvalue is infinitely degenerate. The perturbation of $H_L(B)$ by random Anderson-type potentials V_ω in the weak disorder regime for which $\|V_\omega\| < C_0 B$ has been extensively studied, cf. [3, 4, 5, 6]. It is proved that outside a small interval of size $B/\log B$ about the Landau levels, there are intervals of pure point spectrum with exponentially decaying eigenfunctions. The nature of the spectrum at the Landau levels is unclear. It is now known that there is nontrivial transport near the Landau levels for models on $L^2(\mathbb{R}^2)$ [7]. For a point interaction model on the lattice $\mathbb{Z}^2$, studied in [8], the authors considered the first N Landau levels and proved that there exists an $B_N > 0$ so that if $B > B_N$, then the spectrum of H_ω below the N^{th} Landau level is pure point almost surely and that each Landau level below the N^{th} is infinitely degenerate.

The quantum devices studied with regard to the IQHE may be infinitely extended or finite, but are distinguished by the fact that there is at least one edge, that can be considered infinitely extended, like in the case of the half-plane, or periodic, as in case of an annulus or cylinder. In all cases, the unperturbed Hamiltonian is a nonnegative, self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^2)$ and having the form

$$H_0 = H_L(B) + V_0, \tag{1.2}$$

where V_0 denotes the confining potential forming the edge (we also consider Dirichlet boundary conditions). The existence of an edge profoundly changes the transport and spectral properties of the quantum system. We consider states $\psi \in L^2(\mathbb{R}^2)$ with energy concentration between two successive Landau levels $E_n(B)$ and $E_{n+1}(B)$. We say that such a state ψ carries an *edge current* if the expectation of the y-component of the velocity operator $V_y \equiv (p_y - Bx)$ in the state ψ is nonvanishing. In these two papers, we prove the existence of edge currents carried by these states and provide an explicit lower bound on the strength of the current. This lower bound shows that the edge current persists for all time in that the expectation of the Heisenberg time-dependent current operator $V_y(t) \equiv e^{itH} V_y e^{-itH}$ in the state ψ satisfies the same lower bound for all $t \in \mathbb{R}$. We will also prove that the states that carry edge-currents are well-localized in a neighborhood of the boundary of the region.

Our main results, presented in this paper and its sequel, concern the following geometries and confining potentials.

1. One-Edge Geometries: We study the half-plane case for which the electron is constrained to the right half-plane $x > 0$ by a confining potential V_0 that has either of the two forms:

   (a) Hard Confining Potentials, such as the Sharp Confining Potential: $V_0(x) = \mathcal{V}_0 \chi_{\{x<0\}}(x)$, where $\mathcal{V}_0 > 0$ is a constant, or Dirichlet boundary conditions along the edge $x = 0$.

   (b) Soft Confining Potentials, such as the Polynomial Confining Potential $V_0(x) = \mathcal{V}_0 |x|^p \chi_{\{x<0\}}(x)$, $p \geq 1$, and other rapidly increasing confining potentials.

2. Two-Edge Geometries: We study models for which the electron is confined to the strip $S_L = [-L/2, L/2] \times \mathbb{R}$ by hard or soft confining potentials, such as

   (a) Sharp Confining Potential $V_0(x) = \mathcal{V}_0 \chi_{\{|x|>L/2\}}(x)$.

(b) Parabolic Confining Potential $V_0(x) = \mathcal{V}_0(|x| - L/2)^2 \chi_{|x|>L/2}(x)$

3. Bounded, Two-Edge Geometries: We study models that are topologically a cylinder $\mathbb{R} \times S^1$ with confining potential along the x-direction.

The present paper deals with the first topic of one-edge geometries, and the sequel [9] deals with the second and third topics concerning two-edge geometries.

In addition to these results for straight edge geometries, we show that the results are stable under certain perturbations of the straight edge boundaries. Concerning the hard confining potentials, we note that the lower bounds for the Sharp Confining Potential are uniform with respect to the strength of the confining potential $\mathcal{V}_0$. This means that we can take the limit as the size of the confining potential becomes infinite. As a result, our results extend to the case of Dirichlet boundary conditions along the edges. The various soft confining potentials are discussed in section 6.

Our strategy in the one-edge case is to analyze the unperturbed operator via the partial Fourier transform in the y-variable. We write $\hat{f}(x,k)$ for this partial Fourier transform. This decomposition reduces the problem to a study of the fibered operators of the form

$$h_0(k) = p_x^2 + (k - Bx)^2 + V_0(x), \tag{1.3}$$

acting on $L^2(\mathbb{R})$. Since the effective, nonnegative, potential $V(x;k) = (k - Bx)^2 + V_0(x)$ is unbounded as $|x| \to \infty$, the resolvent of $h_0(k)$ is compact and the spectrum is discrete. We denote the eigenvalues of $h_0(k)$ by $\omega_j(k)$, with corresponding normalized eigenfunctions $\varphi_j(x;k)$, so that

$$h_0(k)\varphi_j(x;k) = \omega_j(k)\varphi_j(x;k), \quad \|\varphi_j(\cdot;k)\| = 1. \tag{1.4}$$

The properties of the eigenvalue maps $k \in \mathbb{R} \to \omega_j(k)$ play an important role in the proofs. These maps are called the *dispersion curves* for the unperturbed Hamiltonian (1.2). The importance of the properties of the dispersion curves comes from an application of the Feynman-Hellmann formula. To illustrate this, let us consider the one-edge geometry of a half-plane with a sharp confining potential that is treated in this paper. It is clear from the form of the effective potential $V(x;k)$ that the dispersion curves are monotone decreasing functions of k, and that $\lim_{k\to+\infty} \omega_n(k) = E_n(B)$, and that $\lim_{k\to-\infty} \omega_n(k) = E_n(B) + \mathcal{V}_0$.

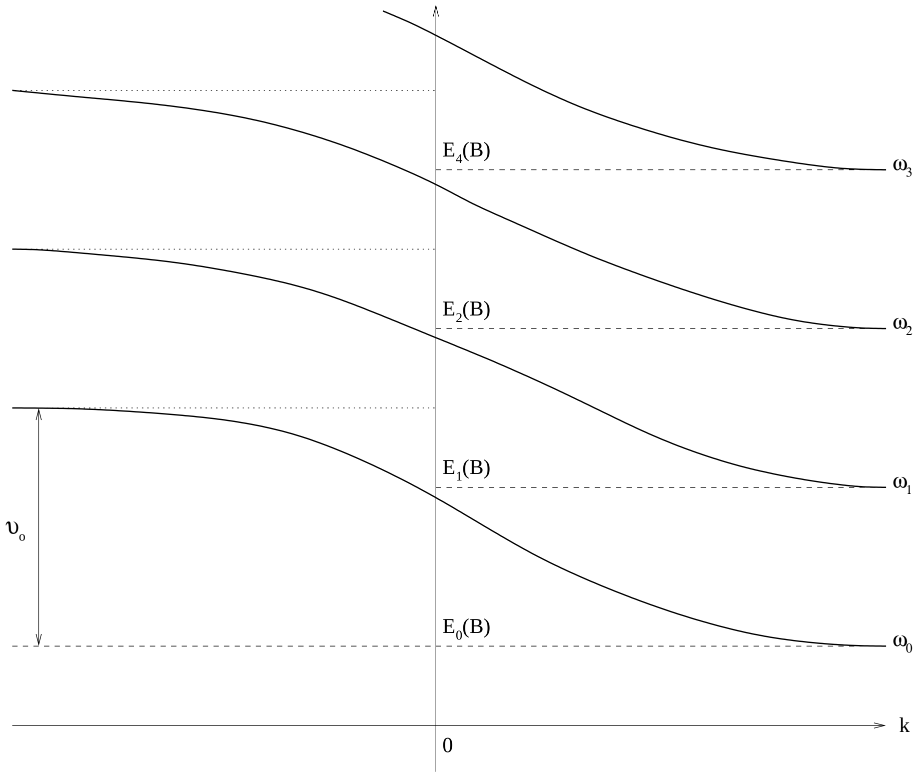

Figure 1 : $\omega_j(k)$ for j=1,2,3,4.

For simplicity, we consider in this introduction a closed interval $\Delta_0 \subset (B, 3B)$ and a normalized wave function ψ satisfying $\psi = E_0(\Delta_0)\psi$, where E_0 is the spectral projection of H_0 associated to Δ_0. Such a function admits a decomposition of the form

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \int_{\omega_0^{-1}(\Delta_0)} e^{iky} \beta_0(k) \varphi_0(x;k)\, dk, \tag{1.5}$$

where the coefficient $\beta_0(k)$ is defined by

$$\beta_0(k) \equiv \langle \hat{\psi}(\cdot, k), \varphi_0(\cdot; k) \rangle, \tag{1.6}$$

with $\hat{\psi}$ denoting the partial Fourier transform given by

$$\hat{\psi}(x,k) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iky} \psi(x,y)\, dy. \tag{1.7}$$

The matrix element of the current operator V_y in such a state is

$$\langle \psi, V_y \psi \rangle = \int_{\mathbb{R}} dx \int_{\omega_0^{-1}(\Delta_0)} dk |\beta_0(k)|^2 (k - Bx) |\varphi_0(x;k)|^2. \tag{1.8}$$

From (1.4) and the Feynman-Hellmann Theorem, we find that

$$\omega_0'(k) = 2 \int_{\omega_0^{-1}(\Delta_0)} dx \ (k - Bx) \ |\varphi_0(x;k)|^2, \tag{1.9}$$

so that we get

$$\langle \psi, V_y \psi \rangle = \frac{1}{2} \int_{\omega_0^{-1}(\Delta)} |\beta_0(k)|^2 \ \omega_0'(k) \ dk. \tag{1.10}$$

It follows from (1.10) that in order to obtain a lower bound on the expectation of the current operator in the state ψ we need to bound the derivative $\omega_0'(k)$ from below for $k \in \omega_0^{-1}(\Delta_0)$. The next step of the proof involves relating the derivative $\omega_0'(k)$ to the trace of the eigenfunction $\varphi_0(x;k)$ on the boundary $x = 0$. For this, we use the formal commutator expression

$$\hat{V}_y(k) \equiv (k - Bx) = \frac{-i}{2B}[p_x, h_0(k)] + \frac{1}{2B} V_0'(x). \tag{1.11}$$

Inserting this into the identity (1.9), we find

$$\begin{aligned} \omega_0'(k) &= 2\langle \varphi_0(\cdot;k), (k - Bx)\varphi_0(\cdot;k)\rangle \\ &= \frac{-i}{2B}\langle \varphi_0(\cdot;k), [p_x, h_0(k)]\varphi_0(\cdot;k)\rangle + \frac{-\mathcal{V}_0}{B}\varphi_0(0;k)^2 \\ &= \frac{-\mathcal{V}_0}{B}\varphi_0(0;k)^2, \end{aligned} \tag{1.12}$$

since the commutator term vanishes by the Virial Theorem. Consequently, we are left with the task of estimating the trace of the eigenfunction along the boundary. Much of our technical work is devoted to obtaining lower bounds on quantities of the form $\mathcal{V}_0 \varphi_n(0;k)^2$, for $n = 0, 1, 2, \ldots$. The situation for the two-edge geometries is more complicated since there is an edge current associated with each edge. This analysis of two-edge geometries is the subject of [9].

Let $H = H_L(B) + V_0 + V_1$ be a perturbation of the one-edge Hamiltonian with spectral family $E(\cdot)$. We consider an energy interval $\Delta_n \subset (E_n(B), E_{n+1}(B))$, and $|\Delta_n|$ small. Roughly speaking, the main result of this paper is a uniform lower bound on the expectation of edge currents in all states with energy localized in the interval Δ_n. We prove that for each $n \in \mathbb{N}$, there exists a finite constant $C_n > 0$ (given precisely below), so that if $\psi \in E(\Delta_n)L^2(\mathbb{R}^2)$, and the perturbation V_1 is such that $\|V_1\|_\infty / B$ is sufficiently small, then

$$|\langle \psi, V_y \psi \rangle| \geq C_n B^{1/2} \|\psi\|^2. \tag{1.13}$$

We note that the order $B^{1/2}$ in (1.13) is optimal as for the unperturbed model, we prove that

$$C_n B^{1/2} \|\psi\|^2 \leq |\langle \psi, V_y \psi \rangle| \leq (1/C_n) B^{1/2} \|\psi\|^2. \tag{1.14}$$

We make two remarks about this result, one concerning the time-dependent theory, and the second concerning the IQHE. First, we remark that the time-independent estimate (1.13) implies that the current persists with at least the same strength for all times provided that the bulk Hamiltonian $H_{bulk} = H_L(B) + V_1$ has a gap in its spectrum between the Landau levels. That is, the estimate (1.13) remains the same if we replace ψ with $\psi_t = e^{-iHt}\psi$, or, equivalently, if we replace the current operator V_y with the Heisenberg current operator $V_y(t) = e^{-iHt} V_y e^{iHt}$. The edge current also remains localized in a neighborhood of size $\mathcal{O}(B^{-1/2})$ near the boundary for all time. Secondly, it has recently been proved that the conductivity corresponding to the edge current, called the *edge conductivity* σ_e, is quantized, and, in fact, equal to the bulk conductivity, σ_b. The edge currents studied in this paper correspond to the edge conductivity and we refer to the papers [10, 11, 12, 13, 14, 15, 16, 17]. For the importance of edge currents in the IQHE, we refer to the papers [14, 18, 19].

1.1 Related Papers

There are several papers on the subject of edge currents for unbounded, one-edge geometries. Macris, Martin, and Pulé [20] studied the half-plane case of one straight edge with *soft* confining potentials. We extend this work proving the existence of edge currents for a large family of soft confining potentials in section 6. Furthermore, we show that we can interpolate between soft and hard confining potentials. DeBièvre and Pulé [21] considered the case of a *hard* confining potential, that is, Dirichlet boundary conditions (DBC). We treat this case in sections 3 and 5 and show that show that one can interpolate between soft and hard confining potentials. The case of DBC was also treated by Fröhlich, Graf, and Walcher [22] who studied non-straight edges. We consider non-straight edges in section 4. As explained in section 5, these papers [20, 21, 22] linked the spectral properties of the one-edge Hamiltonians to the existence of edge currents through the use of the Mourre commutator method. We discuss this thoroughly in section 5. The main interest in spectral properties is due to the fact that these authors prove that under weak perturbations (relative to B) there is absolutely continuous spectrum in the intervals Δ_n. It was pointed out by Exner, Joye, and Kovařík [23] that absolutely continuous spectrum and edge currents can appear when the edge is simply an infinite array of point interactions. These authors studied the Hamiltonian (1.2) for which $V_0(x) = \sum_{j \in \mathbb{Z}} \alpha \delta(x - j)$, and proved that there are bands of absolutely continuous spectra between the Landau levels and that the Landau levels remain infinitely degenerate. More recently, Buchendorfer and Graf [24] developed a scattering theory for edge states in one-edge geometries. These authors show that edge states acquire a phase due to a bend in the boundary relative to a state propagating along a straight boundary. This work has some similarities with the material in section 4.

1.2 Contents

The content of this paper is as follows. Section 2 is devoted the proofs of the edge current estimates for the case of a Sharp Confining Potential and a straight edge. In section 3, we extend these results to the case of Dirichlet boundary conditions along the straight edge. Section 4 is devoted to considering more general boundaries. We introduce the notion of asymptotic edge currents and use scattering theory to prove the stability of these currents. Spectral properties of the Hamiltonians associated with one-edge geometries are studied in section 5 using the Mourre commutator method. In section 6, we extend the results to soft confining potentials. The paper concludes with two appendices. The first appendix in section 7 presents results on the dispersion curves needed in the proofs. The second appendix in section 8, of independent interest, provides explicit pointwise upper and lower bounds on solutions to a certain form of second-order ODEs.

1.3 Acknowledgments

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2 The Straight Edge and a Sharp Confining Potential

In this section, we prove an explicit lower-bound on the edge current formed by a sharp confining potential $V_0(x) = \mathcal{V}_0 \chi_{\{x<0\}}(x)$ along the straight edge $x = 0$. The nonperturbed, one-edge geometry Hamiltonian $H_0 = H_L(B) + V_0$, is a nonnegative, self-adjoint operator on $D(H_L(B))$. We write $E_0(\cdot)$ for the spectral family of H_0. If a classical electron has energy below $\mathcal{V}_0$, then the corresponding classical Hamiltonian describes the dynamics of the particle in the half-plane $x > 0$, the classically allowed region. The complementary region is the classically forbidden region for an electron with energy less than $\mathcal{V}_0$. The edge $x = 0$ reflects the cyclotron orbits of these electrons and causes a net drift of the electron along the edge. This is the origin of the edge current. We will later treat a general family of perturbations V_1, and prove the persistence of edge currents, provided $\|V_1\|_\infty$ is small enough relative to B (and without assuming that V_1 is differentiable as required by some commutator methods). As discussed in section 5, similar results for more restrictive potentials V_1 can be derived from commutator estimates, as obtained by DeBièvre and Pulé [21], and by Fröhlich, Graf, and Walcher [22].

2.1 The Main Results for the Unperturbed Case

Our main result is an explicit lower-bound on the size of the edge current for half-plane in certain states for the unperturbed Hamiltonian H_0. In order to formulate the main theorem, we need to describe these states. Because the edge is straight, we can use the Fourier transform with respect to the y-variable to reduce the problem to a one-dimensional one. The unperturbed operator H_0 admits a partial Fourier decomposition with respect to the y-variable, and the Hilbert space $L^2(\mathbb{R}^2)$ can be expressed as a constant fiber direct integral over $\mathbb{R}$ with fibers $L^2(\mathbb{R})$. For H_0, we write

$$H_0 = \int_{\mathbb{R}}^{\oplus} h_0(k)\ dk, \tag{2.1}$$

where

$$h_0(k) = p_x^2 + (k - Bx)^2 + V_0(x), \quad \text{on} \quad L^2(\mathbb{R}). \tag{2.2}$$

As in section 1, we write $\varphi_j(x;k)$ and $\omega_j(k)$ for the normalized eigenfunctions and the corresponding eigenvalues. The eigenvalues are nondegenerate (cf. section 7) and, consequently, we choose the eigenfunctions φ_j to be real. These eigenfunctions form an orthonormal basis of $L^2(\mathbb{R})$, for any $k \in \mathbb{R}$. Because the map $k \to h_0(k)$ is operator analytic, the simple eigenvalues $\omega_j(k)$ are analytic functions of k. We are interested in states that are energy localized in intervals Δ_n lying between two consecutive Landau levels, that is $\Delta_n \subset (E_n(B), E_{n+1}(B))$. Consider a state ψ having the property that $\psi = E_0(\Delta_n)\psi$. For such a state ψ, we can take the Fourier transform of ψ with respect to y and, using an eigenfunction expansion, write

$$\psi(x,y) = \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{n} \int_{\mathbb{R}} e^{iky}\ \chi_{\omega_j^{-1}(\Delta_n)}(k)\beta_j(k)\varphi_j(x;k)\ dk, \tag{2.3}$$

where the coefficients $\beta_j(k)$ are defined by

$$\beta_j(k) \equiv \langle \hat{\psi}(\cdot,k), \varphi_j(\cdot;k)\rangle, \tag{2.4}$$

where the partial Fourier transform is defined in (1.7). The normalization is such

$$\|\psi\|^2_{L^2(\mathbb{R}^2)} = \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} |\beta_j(k)|^2\ dk. \tag{2.5}$$

Throughout the paper, we will take the interval $\Delta_n \subset (E_n(B), E_{n+1}(B))$ to be given by

$$\Delta_n = [(2n+a)B, (2n+c)B], \quad \text{for} \quad 1 < a < c < 3. \tag{2.6}$$

We can now state the main theorem for the unperturbed, single straight edge Hamiltonian H_0 with a sharp confining potential.

Theorem 2.1 *For $n \geq 0$, let Δ_n be as in (2.6), and suppose that $\mathcal{V}_0 > (2n+3)B$. Let $E_0(\Delta_n)$ be the spectral projection for H_0 and the interval Δ_n. Let $\psi \in L^2(\mathbb{R}^2)$ be a state satisfying $\psi = E_0(\Delta_n)\psi$ with an expansion as in (2.3)–(2.5). Then, for $c - a > 0$ sufficiently small, if $n \geq 1$, so that condition (2.14) is satisfied, we have,*

$$\begin{aligned} -\langle \psi, V_y \psi \rangle \quad \geq \quad & \frac{1}{2^4(n+1)^2[\mathcal{H}^{(n)}]^2} \left(\frac{\pi}{B^7} \right)^{1/2} \sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} |\beta_j(k)|^2 \\ & \times \left(1 - \frac{\omega_j(k)}{\mathcal{V}_0} \right) (\omega_j(k) - E_n(B))^2 (E_{n+1}(B)) - \omega_j(k))^2 \; dk, \end{aligned} \tag{2.7}$$

where the constant $\mathcal{H}^{(n)}$ is defined in (2.39).

Let us note a simplification of the above expression under reasonable conditions. For $k \in \omega_j^{-1}(\Delta_n)$, $j = 0, \ldots, n$, we have

$$(\omega_j(k) - E_n(B))^2 \geq B^2(a-1)^2, \quad (E_{n+1}(B) - \omega_j(k))^2 \geq B^2(3-c)^2. \tag{2.8}$$

Corollary 2.1 *Let us suppose that $\mathcal{V}_0 > (2n+3)B$, for $n \geq 0$, is such that for $k \in \omega_j^{-1}(\Delta_n)$, we have*

$$\left(1 - \frac{\omega_j(k)}{\mathcal{V}_0} \right) > \frac{1}{2}. \tag{2.9}$$

Then, under this condition, the hypotheses of Theorem 2.1, and recalling (2.8), the edge current satisfies the bound

$$-\langle \psi, V_y \psi \rangle \geq \frac{\pi^{1/2}(a-1)^2(3-c)^2}{2^5(n+1)^2[\mathcal{H}^{(n)}]^2} \; B^{1/2} \|\psi\|^2. \tag{2.10}$$

This result shows that any state with energy between $E_n(B)$ and $E_{n+1}(B)$ carries an edge current. However, as the energy approaches a Landau level, the state may delocalize away from the edge.

2.2 Proof of Theorem 2.1.

In order to prove Theorem 2.1, we note that from the representation (2.3), the matrix element of the edge current can be written as

$$\langle \psi, V_y \psi \rangle = \mathcal{M}_n(\psi) + \mathcal{E}_n(\psi), \tag{2.11}$$

where the main term $\mathcal{M}_n(\psi)$ is given by

$$\mathcal{M}_n(\psi) \equiv \sum_{j=0}^{n} \int_{\mathbb{R}} \chi_{\omega_j^{-1}(\Delta_n)}(k) |\beta_j(k)|^2 \langle \varphi_j(\cdot;k), (k - Bx)\varphi_j(\cdot;k)\rangle \; dk, \tag{2.12}$$

and $\mathcal{E}_n(\psi)$ is the error term involving the cross-terms between different Landau levels:

$$\mathcal{E}_n(\psi) \equiv \sum_{j\neq l; j,l=0}^{n} \int_{\mathbb{R}} \chi_{\omega_l^{-1}(\Delta_n)}(k)\, \chi_{\omega_j^{-1}(\Delta_n)}(k) \overline{\beta}_l(k) \beta_j(k) \langle \varphi_l(\cdot;k), (k - Bx)\varphi_j(\cdot;k)\rangle \; dk. \tag{2.13}$$

Concerning this term, we have the following result.

Lemma 2.1 *Suppose $\Delta_n \subset (E_n(B), E_{n+1}(B))$ has the form given in (2.6). Under the conditions described above, if $c - a$ is sufficiently small so that condition (2.14) is satisfied, then the error term (2.13) for the unperturbed problem with any $0 \leq \mathcal{V}_0 < \infty$ is zero: $\mathcal{E}_n(\psi) = 0$.*

Proof. The vanishing of $\mathcal{E}_n(\psi)$ follows from the fact that $\sigma_{jl} \equiv \omega_l^{-1}(\Delta_n) \cap \omega_j^{-1}(\Delta_n) = \emptyset$, for $j \neq l$ and for $|\Delta_n|$ sufficiently small. Each dispersion curve $\omega_j(k)$ is strictly monotone decreasing as follows from the representation (1.12), together with the formula (2.16) in Proposition 2.1 and the bound in Lemma 2.3. Furthermore, due to the simplicity of the spectrum of $h_0(k)$ (see Proposition 7.2) the dispersion curves never intersect. Let us suppose that $\omega_j(k) < \omega_l(k)$, and let k_l^c be the unique point satisfying $\omega_l(k) = (2n + c)B$. Now, it is easy to check that the condition that guarantees that $\sigma_{jl} = \emptyset$ is that

$$((2n + c)B - \omega_j(k_l^c)) > (c - a)B. \tag{2.14}$$

Since the right side of (2.14) can be made small be taking a close to c, whereas the left side is independent of a, this proves the result. □

We note that even when the sets σ_{jl} are nonempty, the eigenfunctions of the reduced Hamiltonians $h_0(k)$ are spatially localized so that the error term $\mathcal{E}_n(\psi)$ is exponentially small.

We therefore have to estimate the main term in (2.11). It is clear that we need to control the matrix element of $\hat{V}_y = (k - Bx)$ in the states $\varphi_j(x;k)$. The following formal commutator expression plays an important role in the calculation of the current in these eigenstates:

$$\hat{V}_y = (k - Bx) \equiv \frac{-i}{2B}[p_x, h_0(k)] + \frac{1}{2B}V_0', \tag{2.15}$$

where V_0' is interpreted in the distributional sense. As a first step, we note the following basic result that follows from analyticity, the Virial Theorem, the existence of $\varphi_j(0;k)$ as proved in Proposition 7.1, and the expression (2.15).

Proposition 2.1 *Let $\varphi_j(x;k)$ be an eigenfunction of $h_0(k)$, with eigenvalue $\omega_j(k)$. We have*

$$\langle \varphi_j(\cdot;k), \hat{V}_y \varphi_j(\cdot;k)\rangle = -\frac{\mathcal{V}_0}{2B}\varphi_j(0;k)^2. \tag{2.16}$$

Recall that the matrix element in (2.16) is equal to $(1/2)\omega_j'(k)$. So the problem is to estimate the slope $\omega_j'(k)$ of the dispersion curves from below for $k \in \omega_j^{-1}(\Delta_n)$, for $j = 1, \ldots, n$. In light of this estimate, the main term of the edge current in (2.11) can be written as

$$\mathcal{M}_n(\psi) \equiv -\frac{1}{2B}\sum_{j=0}^{n}\int_{\omega_j^{-1}(\Delta_n)} |\beta_j(k)|^2 \left(\mathcal{V}_0 \varphi_j(0;k)^2\right) dk. \tag{2.17}$$

Our next step is to obtain a lower bound on the trace of the eigenfunction on the edge, so as to be able to estimate $\mathcal{V}_0\varphi_j(0;k)^2$ from below. This will require several steps.

STEP 1: Eigenfunction Estimate

For the normalized real eigenfunction $\varphi_j(x;k)$, we define, for any $\delta \geq 0$,

$$\eta_j(\delta) \equiv \varphi_j(-\delta;k)^2. \tag{2.18}$$

We now obtain exponential decay results on $\eta_j(\delta)$ as $\delta \to \infty$. An ODE method allows one to obtain a precise form of the prefactor.

Theorem 2.2 *Let $\varphi_j(x;k)$ be the normalized real eigenfunction of $h_0(k)$, defined above, with corresponding eigenvalue $\omega_j(k)$. Then, for any $\delta > 0$, and for all $k \in \mathbb{R}$ so that $0 \leq \omega_j(k) < \mathcal{V}_0$, we have*

$$\eta_j(\delta) \leq \eta_j(0)e^{-\sqrt{2(\mathcal{V}_0-\omega_j(k))}\delta}. \tag{2.19}$$

Proof.

1. The idea of the proof is to obtain a good lower bound on $\eta''_j(\delta)$ and to integrate the result. We refer the reader to appendix 1, Proposition 7.1, on the differentiability of $\varphi_j(x;k)$. The first derivative of $\eta_j(\delta)$ with respect to δ is easily computed $\eta_j'(\delta) = -2\partial_x\varphi(-\delta;k)\ \varphi(-\delta;k)$ whence

$$\eta_j'(\delta) - 2\left[\int_{-\infty}^{-\delta}(\partial_t^2\varphi)(t;k)\varphi(t;k)dt + \int_{-\infty}^{-\delta}(\partial_t\varphi)(t;k)^2 dt\right]. \tag{2.20}$$

We use the eigenvalue equation $h_0(k)\varphi_j = \omega_j(k)\varphi_j$ to re-express $\partial_t^2\varphi_j$ for $t < 0$ as

$$\partial_t^2\varphi_j(t;k) = (k - Bt)^2\varphi_j(t;k) + (\mathcal{V}_0 - \omega_j(k))\varphi_j(t;k). \tag{2.21}$$

Substituting this into (2.20), we obtain,

$$\begin{aligned} -\frac{1}{2}\eta_j'(\delta) &= (\mathcal{V}_0 - \omega_j(k))\int_{-\infty}^{-\delta}\varphi_j(t;k)^2 dt \\ &\quad + \int_{-\infty}^{-\delta}(\partial_t\varphi_j)(t;k)^2 dt + \int_{-\infty}^{-\delta}(k - Bt)^2\varphi_j(t;k)^2\ dt. \end{aligned} \tag{2.22}$$

2. We now take the derivative with respect to δ of the terms in (2.22). This gives

$$\begin{aligned} \frac{1}{2}\eta_j''(\delta) &= (\mathcal{V}_0 - \omega_j(k))\eta(\delta) \\ &\quad + (\partial_x\varphi_j)(-\delta;k)^2 + (k + B\delta)^2\varphi_j(-\delta;k)^2. \end{aligned} \tag{2.23}$$

Since the last two terms on the right of (2.23) are nonnegative, we have

$$\eta_j''(\delta) \geq 2(\mathcal{V}_0 - \omega_j(k))\eta_j(\delta). \tag{2.24}$$

As η_j' obviously converges to zero at infinity, it follows from (2.24) that $\eta_j'(\delta) \leq 0$ for any $\delta \in \mathbb{R}_+$. So multiplying (2.24) by $\eta_j'(\delta)$ and integrating along $[t, +\infty)$ for any $t \geq 0$ yields ${\eta_j'}^2(t) \geq 2(\mathcal{V}_0 - \omega_j(k))\eta_j^2(t)$. By integrating along $[0, \delta]$, for any $\delta \geq 0$, one finally obtains (2.19). □

STEP 2: Harmonic Oscillator Eigenfunction Comparison

It is useful to compare the eigenfunctions of $h_0(k)$ to those of the harmonic oscillator Hamiltonian with no confining potential. The harmonic oscillator Hamiltonian $h_B(k)$ on $L^2(\mathbb{R})$ is defined as

$$h_B(k) \equiv p_x^2 + (k - Bx)^2. \tag{2.25}$$

The eigenvalues of this operator are precisely the Landau energies $E_m(B)$ and are nondegenerate and independent of k. We will denote the real normalized eigenfunctions by $\psi_m(x; k)$. These are given by

$$\psi_m(x; k) = \frac{1}{\sqrt{2^m m!}} \left(\frac{B}{\pi}\right)^{1/4} e^{-\frac{B}{2}(x - k/B)^2} H_m(x\sqrt{B} - (k/\sqrt{B})), \tag{2.26}$$

where $H_m(u)$ is the normalized Hermite polynomial with $H_0(u) = 1$. We expand the eigenfunctions $\varphi_j(x; k)$ in terms of these eigenfunctions

$$\varphi_j(x; k) = \sum_{m=0}^{\infty} \alpha_m^{(j)}(k) \psi_m(x; k), \tag{2.27}$$

where the coefficients are given by

$$\alpha_m^{(j)}(k) = \langle \varphi_j(\cdot; k), \psi_m(\cdot; k) \rangle, \tag{2.28}$$

and satisfy

$$\|\varphi_j(\cdot; k)\|^2 = \sum_{m=0}^{\infty} |\alpha_m^{(j)}(k)|^2 = 1. \tag{2.29}$$

We occasionally suppress the variable k in the notation and write $\alpha_m^{(j)}$ for these coefficients.

Lemma 2.2 *Let $P_n(k)$ be the projection on the eigenspace spanned by the first n eigenfunctions ψ_m of the harmonic oscillator Hamiltonian $h_B(k)$ (2.25).*

Let $\alpha_m^{(j)}$ be the expansion coefficients defined in (2.28). For all $k \in \omega_j^{-1}(\Delta_n)$, with Δ_n as in (2.6), and for all $j = 0, 1, \ldots, n$, we have

$$\sum_{m=0}^{n} |\alpha_m^{(j)}(k)|^2 \geq \frac{1}{2B(n+1)}(E_{n+1}(B) - \omega_j(k)) > 0, \tag{2.30}$$

and

$$|\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle| \geq \frac{1}{2B(n+1)}(\omega_j(k) - E_n(B))(E_{n+1}(B) - \omega_j(k)) > 0. \tag{2.31}$$

Proof.
1. We compute the matrix element $\langle \varphi_j, V_0 \varphi_j\rangle$ using the expansion (2.27),

$$\begin{aligned} \langle \varphi_j, V_0\varphi_j\rangle &= \langle \varphi_j, (h_0(k) - h_B(k))\varphi_j\rangle \\ &= \sum_{m\geq 0}(\omega_j(k) - E_m(B))|\alpha_m^{(j)}(k)|^2, \end{aligned} \tag{2.32}$$

using the normalization (2.29). Rearranging the terms in (2.32), we find

$$\begin{aligned} \sum_{m\leq n}(\omega_j(k) - E_m(B))|\alpha_m^{(j)}(k)|^2 &= \langle \varphi_j, V_0\varphi_j\rangle \\ &\quad + \sum_{m\geq n+1}(E_m(B) - \omega_j(k))|\alpha_m^{(j)}(k)|^2 \\ &\geq (E_{n+1}(B) - \omega_j(k))\left(1 - \sum_{m\leq n}|\alpha_m^{(j)}(k)|^2\right). \end{aligned} \tag{2.33}$$

We now assume that $k \in \omega_j^{-1}(\Delta_n)$ and $j \leq n$. In this case, the coefficient $E_{n+1}(B) - \omega_j(k) > 0$. Moving the second term on the right of (2.33) to the left, we obtain

$$\begin{aligned} (E_{n+1}(B) - \omega_j(k)) &\leq \sum_{m\leq n}(\omega_j(k) - E_m(B) + E_{n+1}(B) - \omega_j(k))|\alpha_m^{(j)}(k)|^2 \\ &= \sum_{m\leq n}(E_{n+1}(B) - E_m(B))|\alpha_m^{(j)}(k)|^2 \\ &\leq 2(n+1)B\left(\sum_{m\leq n}|\alpha_m^{(j)}(k)|^2\right). \end{aligned} \tag{2.34}$$

The result (2.30) follows from (2.34).
2. The calculation of $\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle$, for $k \in \omega_j^{-1}(\Delta_n)$, is similar. We write

$$\begin{aligned}\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle &= \langle \varphi_j(\cdot;k), (h_0(k) - h_B(k))P_n(k)\varphi_j(\cdot;k)\rangle \\ &= \sum_{m\le n} (\omega_j(k) - E_m(B))|\alpha_m^{(j)}(k)|^2 \\ &\ge (\omega_j(k) - E_n(B)) \sum_{m\le n} |\alpha_m^{(j)}(k)|^2 \\ &\ge \frac{1}{2B(n+1)}(\omega_j(k) - E_n(B))(E_{n+1}(B) - \omega_j(k)),\end{aligned} \tag{2.35}$$

where we used (2.30). □

STEP 3: Lower Bound on the Trace

We now use the eigenfunction estimate of Step 1 and the lower bound of Step 2 in order to express the matrix element $\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle$ in terms of the trace of φ_j on the edge.

Lemma 2.3 *Let $\varphi_j(x;k)$ be an eigenfunction of $h_0(k)$, as above, for $0 \le j \le n$. Then, for all $k \in \omega_j^{-1}(\Delta_n)$, we have*

$$\begin{aligned}&\mathcal{V}_0^2 \varphi_j(0;k)^2 \\ &\quad \ge \left(\frac{\pi}{B}\right)^{1/2} \frac{[\mathcal{V}_0 - \omega_j(k)]}{8B^2(n+1)^2[\mathcal{H}^{(n)}]^2}(\omega_j(k) - E_n(B))^2(E_{n+1}(B) - \omega_j(k))^2,\end{aligned} \tag{2.36}$$

where $\mathcal{H}^{(n)}$ is defined in (2.39).

Proof. We use the expansion of φ_j in the eigenfunctions ψ_m and obtain

$$\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle = \sum_{m\le n} \mathcal{V}_0 \alpha_m^{(j)}(k) \int_{-\infty}^0 \varphi_j(x;k)\psi_m(x;k)\, dx. \tag{2.37}$$

To estimate the integral, we use the exponential decay of the eigenfunctions φ_j as given in Theorem 2.2. For $x < 0$, the main eigenfunction decay estimate (2.19) gives

$$\varphi_j(x;k)^2 \le \varphi_j(0;k)^2 e^{-\sqrt{2(\mathcal{V}_0 - \omega_j(k))}|x|}. \tag{2.38}$$

We recall that $\psi_m(x;k)$ is given in (2.26), and define the coefficients

$$\mathcal{H}_m \equiv \sup_{u\in\mathbb{R}} H_m(u)e^{-u^2/2}, \text{ and, } \mathcal{H}^{(n)} \equiv \left(\sum_{m\leq n} \frac{1}{2^m m!} \mathcal{H}_m^2 \right)^{1/2}. \quad (2.39)$$

The integral can be bounded above by

$$\begin{aligned} \left| \int_{-\infty}^{0} \varphi_j(\cdot;k)\psi_m(x;k)\,dx \right| &\leq C_m|\varphi_j(0;k)|\mathcal{H}_m \int_0^\infty e^{-\sqrt{2(\mathcal{V}_0-\omega_j(k))}x}\,dx \\ &\leq \frac{2^{1/2}C_m(B)|\varphi_j(0;k)|\mathcal{H}_m}{\sqrt{(\mathcal{V}_0-\omega_j(k))}}, \end{aligned} \quad (2.40)$$

where $C_m(B) \equiv \left(\frac{B}{\pi}\right)^{1/4}(2^m m!)^{-1/2}$. From (2.37) and (2.40), we get

$$|\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle| \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{2^{1/2}\mathcal{V}_0|\varphi_j(0;k)|}{\sqrt{(\mathcal{V}_0-\omega_j(k))}} \left(\sum_{m\leq n} \frac{1}{\sqrt{2^m m!}} \mathcal{H}_m |\alpha_m^{(j)}(k)| \right). \quad (2.41)$$

Applying the Cauchy-Schwarz inequality to the sum in (2.41), and recalling the normalization (2.29), we find that

$$|\langle \varphi_j(\cdot;k), V_0 P_n(k)\varphi_j(\cdot;k)\rangle| \leq \left(\frac{B}{\pi}\right)^{1/4} \frac{2^{1/2}\mathcal{V}_0|\varphi_j(0;k)|\mathcal{H}^{(n)}}{[\mathcal{V}_0-\omega_j(k)]^{1/2}}. \quad (2.42)$$

We square expression (2.42), and use the bound (2.31) in Lemma 2.4, to obtain the result (2.36). □

The proof of Theorem 2.1 now follows directly from the expression for the main term $\mathcal{M}_n(\psi)$ in (2.17) and the lower bound for the expression $\mathcal{V}_0\varphi_j(0;k)^2$ given in Lemma 2.3. Corollary 2.1 follows directly from the lower bound on the main term.

2.3 Perturbation Theory for the Straight Edge

We now consider the perturbation of H_0 by a bounded potential $V_1(x,y)$. We prove that the lower bound on the edge current is stable with respect to these perturbations provided $\|V_1\|_\infty$ is not too large compared with B. Let Δ_n be as in (2.6). We consider a larger interval $\tilde{\Delta}_n$, containing Δ_n, with the same midpoint $E_n = (2n + (a+c)/2)B \in \Delta_n$, and of the form

$$\tilde{\Delta}_n = [(2n+\tilde{a})B, (2n+\tilde{c})B], \quad \text{for } 1 < \tilde{a} < a < c < \tilde{c} < 3. \quad (2.43)$$

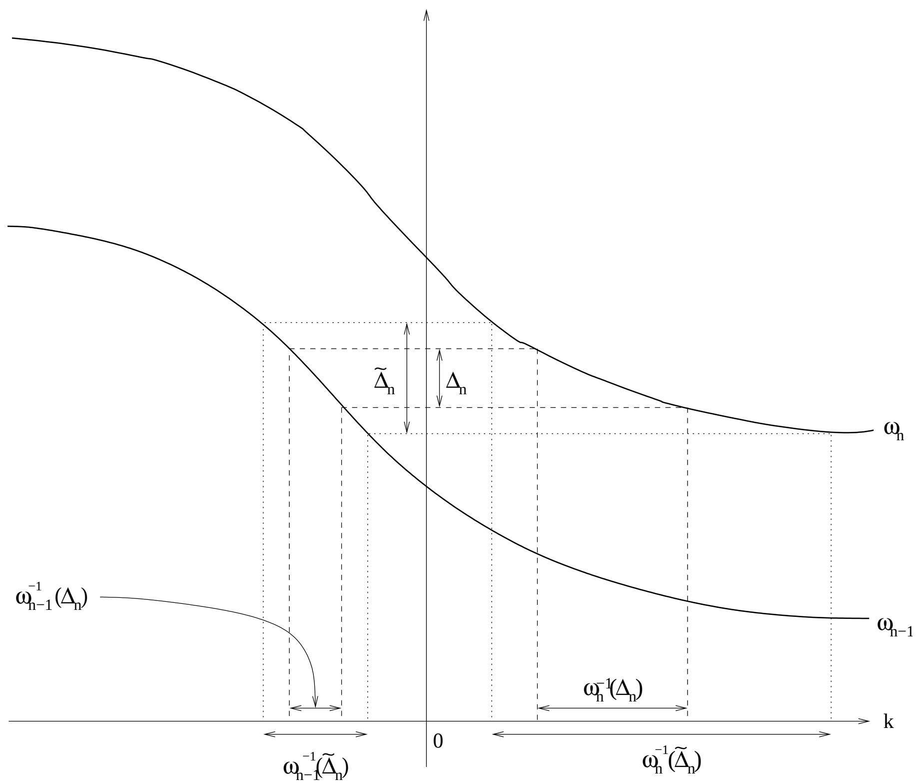

Figure 2 : spectral intervals $\omega_j^{-1}(\Delta_n)$ and $\omega_j^{-1}(\tilde{\Delta}_n)$, j=n−1,n.

In this perturbation argument, we calculate the velocity V_y in states $\psi \in E(\Delta_n)L^2(\mathbb{R}^2)$ that are close to states in $E_0(\tilde{\Delta}_n)L^2(\mathbb{R}^2)$. This closeness is measured by the constant $\kappa > 0$ that we now define. First, we choose the constants $\tilde{a}$ and $\tilde{c}$ in (2.43) so that $\tilde{c} - \tilde{a}$ is small enough for Theorem 2.1 to hold for states in $E_0(\tilde{\Delta}_n)L^2(\mathbb{R}^2)$. Next, we choose a constant $B_n > 0$ large enough and the constants a and c, with $c - a$ small enough, so that for all $B > B_n$, the constant κ defined by

$$\kappa^2 \equiv \left(1 - \left(\frac{2}{\tilde{c} - \tilde{a}} \right)^2 \left(\frac{c-a}{2} + \frac{\|V_1\|_\infty}{B} \right)^2 \right), \tag{2.44}$$

satisfies $0 < \kappa \leq 1$.

Theorem 2.3 *Let $V_1(x, y)$ be a bounded potential and let $E(\Delta_n)$ be the spectral projection for $H = H_0 + V_1$ and the interval Δ_n as in (2.6). Let*

$\psi \in L^2(\mathbb{R}^2)$ be a state satisfying $\psi = E(\Delta_n)\psi$. Let $\phi \equiv E_0(\tilde{\Delta}_n)\psi$ and $\xi \equiv E_0(\tilde{\Delta}_n^c)\psi$, so that $\psi = \phi + \xi$. Under the conditions given above on $a, c, \tilde{a}, \tilde{c}$, and for $B > B_n$, the constant κ, defined in (2.44), satisfies $0 < \kappa \leq 1$ and we have

$$\|\phi\| \geq \kappa \|\psi\|. \tag{2.45}$$

Furthermore, we have the lower bound

$$-\langle \psi, V_y \psi \rangle \geq B^{1/2} \kappa^2 \ (C_n (3 - \tilde{c})^2 (\tilde{a} - 1)^2 - F(n, \|V_1\|/B)) \ \|\psi\|^2, \tag{2.46}$$

where the constants are defined by

$$C_n = \frac{\pi^{1/2}}{2^5 (n+1)^2 [\mathcal{H}^{(n)}]^2}, \tag{2.47}$$

and

$$\begin{aligned} F(n, \|V_1\|_\infty / B) \ &= \ (1 - \kappa^2)^{1/4} \left(2n + c + \frac{\|V_1\|_\infty}{B} \right)^{1/2} \left(2 + \sqrt{1 - \kappa^2} \right) \\ &\quad + C_n (1 - \kappa^2)(3 - \tilde{c})^2 (\tilde{a} - 1)^2. \end{aligned} \tag{2.48}$$

If we suppose that $\|V_1\|_\infty < \mu_0 B$, then for a fixed level n, if $c - a$ and μ_0 are sufficiently small (depending on $\tilde{a}$, $\tilde{c}$, and n), there is a constant $D_n > 0$ so that for all B, we have

$$-\langle \psi, V_y \psi \rangle \geq D_n \kappa^2 B^{1/2} \|\psi\|^2. \tag{2.49}$$

Proof. With reference to the definitions (2.6) and (2.43), we write the function ψ as

$$\psi = E_0(\tilde{\Delta}_n)\psi + E_0(\tilde{\Delta}_n^c)\psi \equiv \phi + \xi. \tag{2.50}$$

We then have

$$\langle \psi, V_y \psi \rangle = \langle \phi, V_y \phi \rangle + 2 Re \langle \phi, V_y \xi \rangle + \langle \xi, V_y \xi \rangle. \tag{2.51}$$

The result follows from Theorem 2.1 provided we have a good bound on $\|\xi\|$ and on $\|V_y \xi\|$. We first note that

$$\begin{aligned} \|\xi\| \ &\leq \ \|E_0(\tilde{\Delta}_n^c)(H_0 - E_n)^{-1}(H - E_n)\psi\| + \|E_0(\tilde{\Delta}_n^c)(H_0 - E_n)^{-1} V_1 \psi\| \\ &\leq \ \left(\frac{2}{\tilde{c} - \tilde{a}} \right) \ \left(\frac{(c - a)}{2} + \frac{\|V_1\|}{B} \right) \|\psi\|. \end{aligned} \tag{2.52}$$

The bound (2.45) follows from (2.52) and the orthogonality of ϕ and ξ. Similarly, we find that

$$\begin{aligned} \|V_y \xi\|^2 &\leq \langle \xi, H_0 \xi \rangle \leq |\langle \psi, H \xi \rangle| + \|V_1\| \; \|\xi\| \; \|\psi\| \\ &\leq ((2n+c)B + \|V_1\|) \; \|\xi\| \; \|\psi\|. \end{aligned} \tag{2.53}$$

Combining (2.52) and (2.53), we obtain

$$|\langle \xi, V_y \xi \rangle| \leq \left(\frac{2}{\tilde{c} - \tilde{a}} \right)^{3/2} B^{1/2} \left(\frac{(c-a)}{2} + \frac{\|V_1\|}{B} \right)^{3/2} \left(2n + c + \frac{\|V_1\|}{B} \right)^{1/2} \|\psi\|^2, \tag{2.54}$$

and

$$|\langle \phi, V_y \xi \rangle| \leq \left(\frac{2}{\tilde{c} - \tilde{a}} \right)^{1/2} B^{1/2} \left(\frac{(c-a)}{2} + \frac{\|V_1\|}{B} \right)^{1/2} \left(2n + c + \frac{\|V_1\|}{B} \right)^{1/2} \|\psi\|^2. \tag{2.55}$$

The lower bound on the main term in (2.51) follows from (2.10) and (2.43),

$$\begin{aligned} -\langle \phi, V_y \phi \rangle &\geq \left(\frac{\pi^{1/2} (\tilde{a} - 1)^2 (\tilde{c} - 3)^2}{2^5 (n+1)^2 [\mathcal{H}^{(n)}]^2} \right) B^{1/2} \left(\sum_{j=0}^{n} \int_{\omega_j^{-1}(\tilde{\Delta}_n)} |\beta_j(k)|^2 \; dk \right) \\ &= \left(\frac{\pi^{1/2} (\tilde{a} - 1)^2 (\tilde{c} - 3)^2}{2^5 (n+1)^2 [\mathcal{H}^{(n)}]^2} \right) B^{1/2} (\|\psi\|^2 - \|\xi\|^2). \end{aligned} \tag{2.56}$$

Combining this lower bound (2.56), with the estimate on $\|\xi\|$ in (2.52), and the bounds (2.53)–(2.55), we find (2.46) with the constants (2.47) and (2.48). This completes the proof. $\Box$

We remark that if the state $\psi \in E(\Delta_n) L^2(\mathbb{R}^2)$ has the property that the corresponding $\phi = 0$, then the right side of (2.56) is zero. It follows from (2.45), however, that if the interval Δ_n is small enough, and if the magnetic field is large enough, then this cannot happen.

2.4 Localization of the Edge Current

It follows from the calculations done above that the edge current carried by states ψ of the unperturbed Hamiltonian H_0 satisfying $\psi = E_0(\Delta_n)\psi$ are localized within a region of size $\mathcal{O}(B^{-1/2})$ near the edge $x = 0$. This corresponds to the classical cyclotron radius. This is made precise in the following theorem.

Theorem 2.4 *Let ψ be a normalized edge-current carrying state, i.e. $\psi = E_0(\Delta_n)\psi$, with $\|\psi\| = 1$. We assume that the interval Δ_n as in (2.6) satisfies $|\Delta_n|/B$ small, and that $\mathcal{V}_0 > (2n+3)B$, as in Theorem 2.3. Then, for any level n and any real numbers $\alpha > -1/2$ and $\beta > 0$, there are three constants $B_{n,\alpha,\beta} > 0$, $C_{n,\alpha,\beta} > 0$, and $K_{n,\alpha,\beta} > 0$, independent of B, such that*

$$\int_{\mathbb{R}} dy \int_{\mathbb{R}\setminus[-B^{-\beta},B^{\alpha}]} dx \; |\psi(x,y)|^2 \leq C_{n,\alpha,\beta} e^{-K_{n,\alpha,\beta}B^{2\alpha+1}}, \tag{2.57}$$

for all $B \geq B_{n,\alpha,\beta}$ and $\mathcal{V}_0 \geq (2n+c)B + B^{2(2\alpha+\beta+1)}$.

Proof. Set $I_{\alpha,\beta} = [-B^{-\beta}, B^{\alpha}]$. In light of the expansion (2.3)–(2.5), and the normalization of ψ, we have

$$\int_{\mathbb{R}} dy \int_{\mathbb{R}\setminus I_{\alpha,\beta}} dx \; |\psi(x,y)|^2 = \sum_{j=1}^{n} \int_{\omega_j^{-1}(\Delta_n)} dk |\beta_j(k)|^2 \int_{\mathbb{R}\setminus I_{\alpha,\beta}} dx |\varphi_j(x;k)|^2. \tag{2.58}$$

Hence, it suffices to prove that the integrals

$$\int_{-\infty}^{-B^{-\beta}} \varphi_j(x;k)^2 \; dx, \text{ and } \int_{B^{\alpha}}^{+\infty} \varphi_j(x;k)^2 \; dx, \tag{2.59}$$

are bounded above as in (2.57) for all $j = 0, 1, \ldots, n$ and $k \in \omega_j^{-1}(\Delta_n)$. The proof consists in four steps.

Step 1. For all $\delta > 1/2$ there is a constant $\tilde{B}_{n,\delta} > 0$ such that we have

$$\omega_j^{-1}(\Delta_n) \subset (-\infty, B^{\delta}), \text{ for all } B \geq \tilde{B}_{n,\delta}. \tag{2.60}$$

To see this, we consider a $C^2(\mathbb{R})$ function J satisfying $J(x) = 0$ for $x \leq 0$, and $J(x) = 1$ for $x \geq B^{-1/2}$, with $\|J'\|_\infty \leq C_1 B^{1/2}$, and $\|J''\|_\infty \leq C_2 B$, for two finite constants $C_1, C_2 > 0$. For all $k \in \mathbb{R}$, $J\psi_n(.;k)$, where $\psi_n(x;k)$ is given in (2.26), belongs to the domain of $h_0(k)$, and we have

$$\begin{aligned} (h_0(k) - (2n+1)B)J(x)\psi_n(x;k) &= [h_0(k), J]\psi_n(x;k) \\ &= -2iJ'(x)\psi_n'(x;k) - J''(x)\psi_n(x;k), \end{aligned}$$

through a direct computation. Moreover the function J' being supported in $[0, B^{-1/2}]$, it follows from this that

$$\begin{aligned} &\|(h_0(k) - (2n+1)B)J\psi_n(.;k)\| \\ \leq\; &2C_1\sqrt{B}\|\chi_B\psi_n'(.;k)\| + C_2 B\|\chi_B\psi_n(.;k)\|, \end{aligned} \tag{2.61}$$

where χ_B is the characteristic function of $[0, B^{-1/2}]$. Next, for all $k \geq B^\delta$ (2.26) assures us there are three constants $B'_{n,\delta} > 0$, $C'_{n,\delta} > 0$ and $K'_{n,\delta} > 0$ such that

$$\|\chi_B \psi_n(.;k)\| + \|\chi_B \psi'_n(.;k)\| \leq C'_{n,\delta} \mathrm{e}^{-K'_{n,\delta} B^{2\delta-1}}, \text{ for } B \geq B'_{n,\delta}.$$

This, combined with (2.61) show that $|\omega_n(k) - (2n+1)B|$ can be made smaller than $(a-1)B$ by taking B sufficiently large. This proves (2.60).

Step 2. Let γ be in $(-1/2, +\infty)$. The normalization condition $\|\varphi_j(.;k)\| = 1$ involving $\int_{B^\gamma/2}^{B^\gamma} \varphi_j^2(x;k)dx \leq 1$, there is necessarily some $x_0 = x_0(B,\gamma)$ in $[B^\gamma/2, B^\gamma]$ such that

$$\varphi_j(x_0;k) \leq \left(\frac{B^\gamma}{2}\right)^{-1/2} \leq 2B^{1/4}. \tag{2.62}$$

In light (2.60) we may also find δ in $(1/2, \gamma+1)$ and $B''_{n,\gamma} > 0$ such that this x_0 together with all $k \in \omega_j^{-1}(\Delta_n)$ are bounded above by B^δ for all $B \geq B''_{n,\gamma}$. As a consequence we have $W_j(x;k) = (Bx-k)^2 - \omega_j(k) \geq B^2(x-x_0)^2 > 0$ and $W'_j(x;k) = 2B^2(x - k/B) > 0$ for all $x > x_0$ and $B \geq B''_{n,\gamma}$, and hence

$$\varphi_j(x;k) \leq \varphi_j(x_0;k)\mathrm{e}^{-B/2(x-x_0)^2}, \text{ for } x \geq x_0 \text{ and } B \geq B''_{n,\gamma},$$

by Lemma 8.3 in Appendix 2. Bearing in mind (2.62) and recalling that $x_0 \leq B^\gamma$, this entails

$$\varphi_j(x;k) \leq 2B^{1/4}\mathrm{e}^{-B/2(x-B^\gamma)^2}, \text{ for } x \geq B^\gamma \text{ and } B \geq B''_{n,\gamma}. \tag{2.63}$$

Step 3. Let α be in $(-1/2, +\infty)$ and set $\gamma = (\alpha - 1/2)/2$. We insert (2.63) in the second integral in (2.59) and get

$$\int_{B^\alpha}^{+\infty} \varphi_j(x;k)^2 dx \leq P_{n,\alpha}(B)\mathrm{e}^{-B/2(B^\alpha - B^\gamma)^2}, \text{ for } B \geq B''_{n,\gamma},$$

where $P_{n,\alpha}(B)$ is a polynomial function of B. This yields that there are three constants $B_{n,\alpha} > 0$, $C_{n,\alpha} > 0$ and $K_{n,\alpha} \in (0,1)$ such that

$$\int_{B^\alpha}^{+\infty} \varphi_j(x;k)^2 dx \leq C_{n,\alpha}\mathrm{e}^{-K_{n,\alpha}B^{2\alpha+1}}, \text{ for } B \geq B_{n,\alpha}. \tag{2.64}$$

Step 4. We turn now to estimating the first integral in (2.59) for some $\beta > 0$. As above we refer to the normalization condition $\|\varphi_j(.;k)\| = 1$ to justify the existence of some $x_1 \in (-B^{-\beta}/2, 0)$ satisfying

$$\varphi_j(x_1;k) \le \sqrt{2}B^{\beta/2}. \tag{2.65}$$

Next $\varphi_j(.;k)$ being solution of the Schrödinger equation $\varphi_j''(x;k) = W_j(x;k)\varphi_j(x;k)$ we choose $\mathcal{V}_0 > (2n+c)B$ so that $W_j(x;k) = (Bx-k)^2 + V_0(x) - \omega_j(k) > 0$ for $x < 0$, and apply Lemma 8.3 in Appendix 2 once more. We get:

$$\varphi_j(x;k) \le \varphi_j(x_1;k)\mathrm{e}^{\sqrt{\mathcal{V}_0-(2n+c)B}(x-x_1)}, \text{ for } x \le x_1.$$

Since $x_1 \ge -B^{-\beta}/2$, this, together with (2.65), entail

$$\int_{-\infty}^{-B^{-\beta}} \varphi_j(x;k)^2 dx \le \mathrm{e}^{-B^{2\alpha+1}},$$

for all $\alpha > -1/2$, $B \ge 1$ and $\mathcal{V}_0 > (2n+c)B + B^{2(2\alpha+\beta+1)}$. Now (2.57) follows from this and from (2.64). □

We now extend this result to the perturbed case. We assume that the conditions guaranteeing the existence of edge current-carrying states for the perturbed Hamiltonian are satisfied. In particular, this means that the perturbation V_1 satisfies a bound $\|V_1\|_\infty < \nu_0 B$, and that $\tilde{c}-\tilde{a}$ is small enough so that $\tilde{\Delta}_n$ lies in the spectral gap of the bulk Hamiltonian $H_{bulk} = H_L(B) + V_1$ in the interval $(E_n(B), E_{n+1}(B))$. We refer the reader to [3, 5] for a discussion of the properties of H_{bulk}. Under these conditions, the edge current for the perturbed Hamiltonian remains close to the wall for all time in a strip of width $B^{-\alpha}$, for any $\alpha < 1/2$, essentially the cyclotron radius. For any $0 < L_0 < \infty$, we define a spatial truncation function $0 \le J_0 \le 1$ to be $J_0(x) = 0$, for $x < L_0$ and $J_0(x) = 1$ for $x > L_0 + 1$.

Theorem 2.5 *Consider the perturbed operator $H = H_0 + V_1$ with $\|V_1\|_\infty < \nu_0 B$, for some constant $0 < \nu_0 < \infty$. Let $\Delta_n \subset \tilde{\Delta}_n = [(2n+\tilde{a})B, (2n+\tilde{c})B]$ lie in the spectral gap of the bulk Hamiltonian $H_{bulk} = H_L(B) + V_1$ in $(E_n(B), E_{n+1}(B))$. Let $\psi = E(\Delta_n)\psi \in L^2(\mathbb{R}^2)$ be an edge current carrying state so that the results of Theorem 2.3 hold true. In particular, we assume that ν_0 and that $\tilde{c} - \tilde{a}$ are small enough so that the lower bound (2.49) is valid. Then, for any level n, and for any $0 < \alpha < 1/2$, there exist constants*

$0 < C_n, K_n < \infty$, *independent of* B, *so that for a strip of width* $L_0 = B^{-\alpha}$, *we have*

$$\|J_0 \psi\| \leq C_n e^{-K_n B^{1/2-\alpha}}. \tag{2.66}$$

Proof. The method of proof is similar to that given in [25]. The resolvent formula for H_{bulk} and H gives

$$R(z) = R_{bulk}(z) - R_{bulk}(z) V_0 R(z). \tag{2.67}$$

Let $0 \leq f \leq 1$ be a smooth, nonnegative function with $f_{|\Delta_n} = 1$ and supp $f \subset \tilde{\Delta}_n$. Then, we can write $\psi = f(H)\psi$. We use the Helffer-Sjöstrand formula for the operator $f(H)$, cf. [25] or [26]. Let $\tilde{f}$ be an almost analytic extension of f into a small complex neighborhood of $\tilde{\Delta}_n$ that vanishes of order two as $\mathrm{Im}\,(z) \to 0$. The Helffer-Sjöstrand formula for $f(H)$ is

$$f(H) = \frac{-1}{\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{f}(z)(H - z)^{-1} \; dx \; dy. \tag{2.68}$$

Note that since the support of f lies in the spectral gap of H_{bulk}, formula (2.68) shows that $f(H_{bulk}) = 0$. Then, by the resolvent formula (2.67), and the Helffer-Sjöstrand formula (2.68), we can write

$$\begin{aligned} J_0 \psi &= J_0 f(H) \psi \\ &= \frac{-1}{\pi} \int_{\mathbb{C}} \partial_{\overline{z}} \tilde{f}(z) J_0 R_{bulk}(z) V_0 R(z) \; dx \; dy. \end{aligned} \tag{2.69}$$

The distance between the supports of the confining potential V_0 and the localization function J_0 is $0 < L_0 < \infty$. An application of the Combes-Thomas method to Landau Hamiltonians as presented, for example, in [3], results in the following bound for the operator $J_0 R_{bulk}(z) V_0$ for z in the resolvent set of H_{bulk}. There are constants $0 < C_1, C_2 < \infty$ so that

$$\|J_0 R_{bulk}(z) V_0\| \leq \frac{C_1}{d(\sigma(H_{bulk}), z)} e^{-C_2 B^{1/2} L_0}. \tag{2.70}$$

The distance $d(\sigma(H_{bulk}), z)$ is given by the minimum of the distance from the larger interval $\tilde{\Delta}_n$ to the band edges of the spectrum of H_{bulk} at $E_n(B) + \|V_1\|_\infty$ and $E_{n+1}(B) - \|V_1\|_\infty$. Consequently, if $L_0 = B^{-\alpha}$, for $\alpha < 1/2$, we obtain the result. □

3 The Straight Edge and Dirichlet Boundary Conditions

We note that the lower bounds on the edge currents in Theorems 2.1 and 2.3 are independent of the size of the confining potential barrier $\mathcal{V}_0$, provided $\mathcal{V}_0 >> E_{n+1}(B)$. This indicates that these lower bounds should remain valid in the limit $\mathcal{V}_0 \to \infty$. This limit formally corresponds to Dirichlet boundary conditions along the edge at $x = 0$. In this section, we use the results of section 2.1 and 2.3 to prove lower bounds on the edge current with Dirichlet boundary conditions (DBC) along $x = 0$. DeBièvre and Pulé [21] and Fröhlich, Graf, and Walcher [22] both considered the Landau Hamiltonian with Dirichlet boundary conditions along the edge $x = 0$ in their articles. Both groups proved the existence of edge currents using the commutator method described in section 5. DeBièvre and Pulé [21] avoid the minor technical difficulty encountered by Fröhlich, Graf, and Walcher [22] due to the nonselfadjointness of p_x on a half line by using y as a conjugate operator. We provide an alternate proof of the existence of edge currents in the hard boundary (DBC) case here that does not use commutator estimates.

We denote the Landau Hamiltonian $H_L(B)$ on the space $L^2([0,\infty) \times \mathbb{R})$ with Dirichlet boundary conditions along $x = 0$ by H_0^D. This unperturbed operator admits a direct integral decomposition with respect to the y-variable. We denote by $h_0^D(k)$ the corresponding fibered operator with eigenvalues $\omega_j^D(k)$ and eigenfunctions $\varphi_j^D(x;k)$. These eigenfunctions provide an eigenfunction expansion of any state, as in (2.3), and we denote the coefficients of this expansion by $\beta_j^D(k)$. The eigenfunctions of $h_0^D(k)$ are given explicitly by Whittaker functions. Many properties of the dispersion curves $\omega_j^D(k)$ are derived from the properties and estimates on Whittaker functions, cf. [21]. The perturbed operator is denoted by $H_D \equiv H_0^D + V_1$, on the same Hilbert space. We let $E_0^D(\cdot)$ and $E_D(\cdot)$ denote the corresponding spectral families. As in section 2, the interval $\Delta_n = [(2n+a)B, (2n+c)B]$, with $1 < a < c < 3$.

Theorem 3.1 *Consider the operators H_0^D and $H_D = H_0^D + V_1$, on $\mathcal{H} \equiv L^2([0,\infty) \times \mathbb{R})$, with Dirichlet boundary conditions along $x = 0$. Any state $\psi \in E_D(\Delta_n)\mathcal{H}$ carries an edge current satisfying the lower bounds (2.46), with the same constants (2.47)–(2.48), provided $(c - a)$ and $\|V_1\|_\infty / B$ are sufficiently small as stated there.*

We prove this theorem through a perturbation argument comparing H_0^D

on $L^2([0,\infty)\times\mathbb{R})$ with $H_0 = H_L(B) + V_0$ acting on $L^2(\mathbb{R}^2)$ in the large $\mathcal{V}_0$ regime. In this regime of very large $\mathcal{V}_0$, the behavior of eigenfunctions with eigenvalues in a fixed energy interval for $x < 0$ becomes unimportant. We begin with an estimate on the trace of the eigenfunctions $\varphi_j(x;k)$ of $h_0(k)$ on the line $x = 0$.

Lemma 3.1 *Let $\varphi_j(x;k)$ be a normalized eigenfunction of $h_0(k)$ as in section 2. For any $0 \le j \le n$, and for all $k \in \omega_j^{-1}(\Delta_n)$, we have*

$$0 \le \varphi_j(0;k) \le \left(\frac{2B}{\mathcal{V}_0}\right)^{1/2} [(2n+3)B]^{1/4}. \tag{3.1}$$

In general, for any eigenfunction $\varphi_l(x;k)$, and for any $k \in \mathbb{R}$, we have

$$0 \le \varphi_l(0;k) \le \left(\frac{2B}{\mathcal{V}_0}\right)^{1/2} \omega_l(k)^{1/4} \le \left(\frac{2B}{\mathcal{V}_0}\right)^{1/2} [(2l+1)B + \mathcal{V}_0]^{1/4}. \tag{3.2}$$

Proof. One can choose $\varphi_j(x;k) \ge 0$, for $x < 0$, as discussed in Appendix 1, Proposition 8.1. From Proposition 2.1, and the consequence of the Feynman-Hellman Theorem (1.9)–(1.10), we have

$$\varphi_j(0;k)^2 = -\frac{2B}{\mathcal{V}_0}\langle \varphi_j(\cdot;k), \hat{V}_y(k)\varphi_j(\cdot;k)\rangle = -\frac{B}{\mathcal{V}_0}\omega_j'(k) \ge 0, \tag{3.3}$$

as we recall that $\omega_j'(k) \le 0$. A simple calculation now gives

$$\begin{aligned} |\omega_j'(k)| &= |\langle \varphi_j(\cdot;k), h_0'(k)\varphi_j(\cdot;k)\rangle| \\ &= 2|\langle \varphi_j(\cdot;k), (k-Bx)\varphi_j(\cdot;k)\rangle| \\ &\le 2|\langle \varphi_j(\cdot;k), (k-Bx)^2\varphi_j(\cdot;k)\rangle|^{1/2} \\ &\le 2\omega_j(k)^{1/2} \le 2[(2n+3)B]^{1/2}, \end{aligned} \tag{3.4}$$

by positivity of the operator $h_0(k)$, and the fact that $k \in \omega_j^{-1}(\Delta_n)$. Combining this with (3.3), we obtain the bound (3.1). The bound (3.2) follows from (3.4) and the structure of the dispersion curves. □

We next show how Lemma 3.1 implies the convergence of the dispersion curves $\omega_j(k)$ to $\omega_j^D(k)$ as $\mathcal{V}_0 \to \infty$. We use an estimate on the eigenvalues $\omega_j^D(k)$ of the Dirichlet problem that follows from an estimate in Lemma 2.1

of De Bièvre and Pulé [21]. The explicit properties of the eigenfunctions $\varphi_j(x;k)$ allow one to prove that if $j \neq l$, then there is a finite constant $C_{jl} > 0$ so that

$$|\omega_j^D(k) - \omega_l^D(k)| \geq C_{jl} B, \quad \forall k \in \mathbb{R}. \tag{3.5}$$

Lemma 3.2 *The dispersion curves $\omega_j(k)$ are monotonic increasing functions of $\mathcal{V}_0$. For $\mathcal{V}_0 >> E_{n+1}(B)$, and for $j = 0, \ldots, n$, and for $k \in \omega_j^{-1}(\Delta_n)$, we have*

$$0 \leq \omega_j^D(k) - \omega_j(k) \leq \frac{C_0(n,B)}{\mathcal{V}_0^{1/2}}. \tag{3.6}$$

Proof. The Hamiltonians $h_0(k)$ are analytic operators in the parameter $\mathcal{V}_0$. We use the Feynman-Hellman Theorem to compute the variation of the eigenvalues $\omega_j(k)$ with respect to $\mathcal{V}_0$. This gives

$$\frac{\partial \omega_j}{\partial \mathcal{V}_0}(k) = \int_{\mathbb{R}^-} \varphi_j(x;k)^2 \, dx \geq 0, \tag{3.7}$$

so that the dispersion curves are monotone increasing with respect to $\mathcal{V}_0$. Furthermore, the rate of increase in (3.7) slows as $\mathcal{V}_0 \to \infty$. This follows from the pointwise upper bound on $\varphi_j(x,k)$ restricted to $x \leq 0$. In particular, from (2.38) and the trace estimate (3.1), we have

$$\begin{aligned} 0 \leq \frac{\partial \omega_j}{\partial \mathcal{V}_0}(k) &\leq \varphi_j(0;k)^2 \int_{-\infty}^0 e^{-2\sqrt{(\mathcal{V}_0 - \omega_j(k))}|x|} \, dx \\ &\leq \frac{(2n+3)^{1/2}}{\sqrt{(\mathcal{V}_0 - \omega_j(k))}} \left(\frac{B^{3/2}}{\mathcal{V}_0} \right). \end{aligned} \tag{3.8}$$

This shows that the dispersion curve $\omega_j^D(k)$ is an upper bound on the dispersion curves $\omega_j(k)$. To prove the rate of convergence (3.6), we use the eigenvalue equation

$$-\varphi_j''(x) + (k - Bx)^2 \varphi_j(x) = \omega_j(k) \varphi_j(x), \;\; x > 0 \tag{3.9}$$

and take the inner product in $\mathbb{R}_+$ with the Dirichlet eigenfunction φ_l^D. After integration by parts, and an application of the eigenvalue equation for φ_l^D, one obtains,

$$(\omega_l^D(k) - \omega_j(k)) \langle \varphi_l^D(\cdot;k), \varphi_j(\cdot;k) \rangle = (\varphi_l^D)'(0;k) \varphi_j(0;k). \tag{3.10}$$

The estimate in Lemma 3.1 implies that the left side of (3.10) vanishes as $\mathcal{V}_0 \to \infty$, that is

$$|\omega_l^D(k) - \omega_j(k)| \, |\langle \varphi_l^D(\cdot;k), \varphi_j(\cdot;k)\rangle| \leq |(\varphi_l^D)'(0;k)| \left(\frac{2B}{\mathcal{V}_0}\right)^{1/2} [(2n+3)B]^{1/2}. \tag{3.11}$$

We next show that $|\langle \varphi_j^D(\cdot;k), \varphi_j(\cdot;k)\rangle|$ is uniformly bounded from below as $\mathcal{V}_0 \to \infty$, proving the convergence of the eigenvalues. To show this, let $\chi_\pm$ denote the characteristic functions onto the left and right half lines $(-\infty, 0]$ and $[0, \infty)$, respectively. We first note that

$$\|\varphi_j(\cdot;k)\|^2 = 1 = \|\chi_- \varphi_j(\cdot;k)\|^2 + \|\chi_+ \varphi_j(\cdot;k)\|^2, \tag{3.12}$$

and the upper bound on the eigenfunction φ_j on the negative half-axis (2.38), together with (3.1), imply that $\|\chi_- \varphi_j(\cdot;k)\| \leq C_j \mathcal{V}_0^{-3/4}$, so that

$$\|\chi_+ \varphi_j(\cdot;k)\| \geq 1 - \mathcal{O}(\mathcal{V}_0^{-3/4}), \tag{3.13}$$

as $\mathcal{V}_0 \to \infty$ and $k \in \omega_j^{-1}(\Delta_n)$. Now, for $l \neq j$, it follows from (3.5) and the monotonicity of the dispersion curves in $\mathcal{V}_0$ that

$$|\omega_l^D(k) - \omega_j(k)| \geq |\omega_l^D(k) - \omega_j^D(k)| \geq C_{lj} B. \tag{3.14}$$

So it follows from this (3.14) and from (3.11) that for $l \neq j$

$$\langle \varphi_l^D(\cdot;k), \varphi_j(\cdot;k)\rangle \to 0, \quad \text{as} \quad \mathcal{V}_0 \to \infty. \tag{3.15}$$

If, in addition, the matrix element $\langle \varphi_j^D(\cdot;k), \varphi_j(\cdot;k)\rangle$ also vanished as $\mathcal{V}_0 \to \infty$, this would contradict (3.13) as the family $\{\varphi_l^D(\cdot;k)\}$ is an orthonormal basis. It follows that this matrix element must be bounded from below uniformly in $\mathcal{V}_0$ as $\mathcal{V}_0 \to \infty$. Consequently, the dispersion curves must converge as $\mathcal{V}_0 \to \infty$ with the specified rate. □

The local convergence of the dispersion curves to those for the Dirichlet problem is a key ingredient in proving the convergence of the projection $P_j(k)$, for the eigenvalue $\omega_j(k)$ of $h_0(k)$, to the projector $P_0^D(k)$, for the eigenvalue $\omega_j^D(k)$ of $h_0^D(k)$, when $\mathcal{V}_0$ tends to infinity (with B fixed). The proof relies on the comparison of the resolvents $R_0(z;k) = (h_0(k) - z)^{-1}$ and $R_0^D(z;k) = (h_0^D(k) - z)^{-1}$, as $\mathcal{V}_0 \to \infty$, for $z \in \Gamma_j(\mathcal{V}_0)$, a contour of radius $1/\mathcal{V}_0^{3/8}$ about $\omega_j^D(k)$, for $0 \leq j \leq n$ and $k \in \Sigma_n \equiv \cup_{j=0}^n \omega_j^{-1}(\Delta_n)$. The comparison of the resolvents relies on a formula derived from Green's theorem and various trace estimates. This is rather standard; we refer, for example, to the discussion in [27]. This is the content of the next lemma.

Lemma 3.3 *Let $P_j(k)$, respectively $P_j^D(k)$, for $j = 0, \ldots, n$, be the projection onto the one-dimensional subspace of $h_0(k)$, respectively $h_0^D(k)$, corresponding to the eigenvalue $\omega_j(k)$, respectively $\omega_j^D(k)$. Then, there exists a finite constant $C_1(n, B) > 0$, such that for all $\mathcal{V}_0$ sufficiently large, and uniformly for $k \in (\omega_j^D)^{-1}(\Delta_n) \cup \omega_j^{-1}(\Delta_n)$, we have*

$$\|P_j(k) - P_j^D(k)\| \leq \frac{C_1(n, B)}{\mathcal{V}_0^{1/4}}. \tag{3.16}$$

We outline the main ideas of the proof here and refer the reader to the archived version for complete details [28]. We are concerned with the first $n+1$-eigenvalues $\omega_j(k)$ of $h_0(k)$, for $j = 0, \ldots, n$. We fix $0 \leq j \leq n$, and let $\Gamma_j(\mathcal{V}_0)$ be the circle of radius $1/\mathcal{V}_0^{3/8}$ about $\omega_j^D(k)$. By Lemma 3.2, there is an amplitude $\mathcal{V}_0^* >> 1$ so that $|\omega_j^D(k) - \omega_j(k)| < C_n \mathcal{V}_0^{-1/8}$, and dist $(z, \omega_j(k)) \geq \mathcal{V}_0^{-1/4}$, for $\mathcal{V}_0 > \mathcal{V}_0^*$. Moreover, there exists an index $N(\mathcal{V}_0) >> n$, such that if $l > N(\mathcal{V}_0)$, we have dist $(\omega_l(k), \Gamma_j(\mathcal{V}_0)) > \mathcal{V}_0$. The index $N(\mathcal{V}_0)$ can be chosen to be proportional to $\mathcal{V}_0$ since $\omega_l(k)$ is bounded above by $(2l+1)B + \mathcal{V}_0$.

In order to estimate the difference of the projectors on the left in (3.16), we use the contour representation of the projections in terms of the resolvents so that the difference of the projectors is written as

$$P_j^D(k) - P_j(k) = \frac{1}{2\pi i} \int_{\Gamma_j(\mathcal{V}_0)} (R_0(z;k) - R_0^D(z;k)) \; dz. \tag{3.17}$$

The resolvent formula for the difference of the two resolvents in (3.17) following from Green's theorem is

$$R_0(z;k) - R_0^D(z;k) = R_0(z;k) T_0^* B_0 R_0^D(z;k), \tag{3.18}$$

where T_0 is the trace map $(T_0 u)(x) = u(0)$, and $(B_0 u)(x) = u'(0)$. The trace map is a bounded map from $H^1(\mathbb{R}) \to \mathbb{C}$. Substituting (3.18) into the right side of (3.17), we obtain

$$P_j^D(k) - P_j(k) = \frac{1}{2\pi i} \int_{\Gamma_j(\mathcal{V}_0)} R_0(z;k) T_0^* B_0 R_0^D(z;k) \; dz. \tag{3.19}$$

Due to the simplicity of the eigenvalues, the resolvent $R_0(z;k)$ has the expression

$$R_0(z;k) = \sum_{j=0}^{\infty} \frac{P_j(k)}{\omega_j(k) - z}, \tag{3.20}$$

where $P_j(k)$ projects onto the one-dimensional subspace spanned by $\varphi_j(x;k)$. We have a similar expression for $R_0^D(z;k)$.

In order to exploit the large $\mathcal{V}_0$ regime, we decompose any $\phi \in L^2(\mathbb{R})$ into a piece ϕ^L supported on $(-\infty, 0]$, and its complement: $\phi = \phi^L + \phi^R$. With this decomposition applied to any $\phi, \psi \in L^2(\mathbb{R})$, we write the inner product of the difference of the resolvents as

$$\langle \phi, (R_0(z;k) - R_0^D(z;k))\psi \rangle = \langle \phi^R, (R_0(z;k) - R_0^D(z;k))\psi^R \rangle + \mathcal{E}_{LR}(z;k). \tag{3.21}$$

The mixed error term $\mathcal{E}_{LR}$ has the form

$$\mathcal{E}_{LR}(z;k) = \langle \phi^L, R_0(z;k)\psi^R \rangle + \langle \phi, R_0(z;k)\psi^L \rangle. \tag{3.22}$$

The trace is evaluated using the expansion (3.20) and estimates (3.1)–(3.2). be As a result of some calculations and these estimates, we find that

$$\left| \int_{\Gamma_j(\mathcal{V}_0)} \langle \phi^R, (R_0(z;k) - R_0^D(z;k))\psi^R \rangle \; dz \right| \leq \left(\frac{C_5(n,B)}{\mathcal{V}_0^{1/4}} \right) \|\phi\| \; \|\psi\|. \tag{3.23}$$

Finally, it remains to estimate the error term $\mathcal{E}_{LR}$ in (3.22). This is evaluated by substituting the expansion (3.20) into each inner product of $\mathcal{E}_{LR}$. We then separate each sum into three sets of indices. For the first two sets of indices, $0 \leq j \leq n$, and $n < j \leq N(\mathcal{V}_0)$, the half-line $x < 0$ is in the classically forbidden region for the eigenfunctions $\varphi_j(x;k)$, with $k \in \omega_j^{-1}(\Delta_n)$. For the third set of indices, we have dist $(z, \omega_l(k)) >> C_0\mathcal{V}_0$. For the first two sets of indices, that is for $0 \leq l \leq N(\mathcal{V}_0)$, it follows from section 8 that the eigenfunctions $\varphi_l(x;k)$ satisfy the bound

$$\varphi_l(x;k) \leq \varphi_l(0;k) e^{-\sqrt{V_0 - \omega_l(k)}|x|}, \quad \text{for } x \leq 0. \tag{3.24}$$

Combining these exponential decay estimates with the trace stimates (3.1)–(3.2), we find for the contour integral of the error term $\mathcal{E}_{LR}$,

$$\left| \int_{\Gamma_j(\mathcal{V}_0)} \mathcal{E}_{LR}(z;k) \; dz \right| \leq \frac{C_8(n,B)}{\mathcal{V}_0^{5/8}} \|\phi\| \; \|\psi\|. \tag{3.25}$$

This estimate, and the estimate (3.23) of the main term prove the result (3.16).

Proof of Theorem 3.1. We begin with the unperturbed case. Let $\psi \in L^2(\mathbb{R}^+ \times \mathbb{R})$ satisfy $\psi = E_0^D(\Delta_n)\psi$. We assume that the hypotheses of Lemma 2.1 hold so that there are no cross-terms in the matrix element $\langle \psi, V_y \psi \rangle$. We will use the results of Lemma 3.2 that tell us that $\omega_j(k) \to \omega_j^D(k)$, locally, and that the matrix element $\langle \varphi_j^D(\cdot; k), \varphi(\cdot; k) \rangle \geq D_0$, as $\mathcal{V}_0 \to \infty$. We write

$$\begin{aligned}
-\langle \psi, V_y \psi \rangle &= -\sum_{j=0}^{n} \int_{(\omega_j^D)^{-1}(\Delta_n)} dk |\beta_j^D(k)|^2 \langle \varphi_j^D(\cdot; k), P_j^D(k) \hat{V}_y(k) P_j^D(k) \varphi_j^D(\cdot; k) \rangle \\
&\geq -\sum_{j=0}^{n} \int_{(\omega_j^D)^{-1}(\Delta_n)} dk |\beta_j^D(k)|^2 |\langle \varphi_j^D(\cdot; k), \varphi_j(\cdot; k) \rangle|^2 \\
&\quad \times \langle \varphi_j(\cdot; k), P_j(k) \hat{V}_y(k) P_j(k) \varphi_j(\cdot; k) \rangle - \mathcal{R}(\psi) \\
&\geq -D_0 \sum_{j=0}^{n} \int_{(\omega_j^D)^{-1}(\Delta_n)} dk |\beta_j^D(k)|^2 \langle \varphi_j(\cdot; k), P_j(k) \hat{V}_y(k) P_j(k) \varphi_j(\cdot; k) \rangle \\
&\quad - \mathcal{R}(\psi).
\end{aligned} \tag{3.26}$$

The remainder $\mathcal{R}(\psi)$ is bounded by

$$\mathcal{R}(\psi) \leq 2 \sum_{j=0}^{n} \int_{(\omega_j^D)^{-1}(\Delta_n)} dk |\beta_j^D(k)|^2 \left\{ |\langle \varphi_j^D(\cdot; k), (P_j^D(k) - P_j(k)) \hat{V}_y(k) P_j^D(k) \varphi_j^D(\cdot; k) \rangle| \right\}. \tag{3.27}$$

The main term in (3.26) is bounded from below as in Theorem 2.1. Estimates on the difference of the spectral projectors given in Lemma 3.3 establish the appropriate bounds on the remainder $\mathcal{R}(\psi)$. This proves the theorem for the unperturbed case. The perturbation theory of section 2.2 now applies in the same manner as in that section. □

4 One-Edge Geometries with More General Boundaries

The previous results were based on the exact calculations for the unperturbed case due to the possibility of taking the partial Fourier transform.

Fröhlich, Graf, and Walcher [22] considered more general one-edge geometries for which the boundary satisfies some mild regularity conditions. We first review these results, and then present some new results based on the notion of the *asymptotic velocity of edge currents* coming from scattering theory. These results apply to a very general class of perturbations of the half-plane geometry.

Fröhlich, Graf, and Walcher [22] studied one-edge, simply connected, unbounded regions $\Omega \subset \mathbb{R}^2$, with a piecewise C^3-boundary. The boundary must satisfy some additional geometric conditions so that the edge does not asymptotically become parallel to itself so that the region resembles a two-edge geometry near infinity. If this occurs, the interaction of the classical trajectories in different directions may cancel each other. The authors consider the unperturbed Hamiltonian H_0^D which is the Landau Hamiltonian on Ω with Dirichlet boundary conditions on $\partial\Omega$. The main theorem of [22] is the following.

Theorem 4.1 *Assume that the region Ω satisfies the geometric conditions discussed above and that the perturbation $V_1 \in L^\infty(\mathbb{R}^2)$. Let $E/B \notin 2\mathbb{N}+1$ and suppose that B is taken sufficiently large so that $\|V_1\|_\infty / B$ is sufficiently small. Then, the spectrum of $H_\Omega^D = H_0^D + V_1$ is absolutely continuous near E.*

As in the work of DeBièvre and Pulé [21], and as we discuss in section 5, Fröhlich, Graf, and Walcher construct a conjugate operator for the Hamiltonian H_Ω^D on the region Ω. They prove that the commutator, when spectrally localized to a small interval of energies around E, has a strictly positive lower bound. Mourre theory [30] then implies the existence of absolutely continuous spectrum near E. The Dirichlet boundary conditions on $\partial\Omega$ cause some technical complications as p_x is not self-adjoint on any domain. The conjugate operator is a quantization of a linearization of the classical guiding center trajectory for the classical electron orbit.

We introduce another notion to the study of geometrically perturbed regions and use it to prove the persistence of edge currents. The *asymptotic velocity* is defined for any pair of self-adjoint Schrödinger operators (H_0, H) for which the wave operators exist. The (global) wave operators $\Omega_\pm$ for the pair (H_0, H) are defined by

$$\Omega_\pm \equiv s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} E_{ac}(H_0), \tag{4.1}$$

where $E_{ac}(H_0)$ is the projection onto the absolutely continuous spectral subspace for H_0. When the wave operators exist, the range is contained in the absolutely continuous spectral subspace of H, and the wave operators are partial isometries between these spectral subspaces. We will use the local wave operators $\Omega_\pm(\Delta)$ obtained by replacing $E_{ac}(H_0)$ by the projector $E_0(\Delta)$ for H_0 and an interval Δ in the absolutely continuous subspace of H_0. The asymptotic velocity is defined for any component of the velocity observable. We are interested in velocity asymptotically in the y-direction and for states with energy in an interval Δ. We define this to be

$$V_y^\pm(\Delta) \equiv \Omega_\pm(\Delta) V_y \Omega_\pm^*(\Delta). \tag{4.2}$$

We note that when H_0 commutes with V_y, and the local wave operators exist, the local asymptotic velocity is obtained by the limit

$$V_y^\pm(\Delta) \equiv s - \lim_{t \to \pm\infty} e^{itH} E_0(\Delta) V_y E_0(\Delta) e^{-itH}. \tag{4.3}$$

In the context of potential scattering, we refer to the book of Derezinski and Gérard [29] for a complete discussion of the *asymptotic velocity*.

We consider the geometric perturbation of the straight, one-edge geometry obtained by perturbing the boundary confining potential V_0. We recall that a *sharp confining potential* V_0 is a constant multiple $\mathcal{V}_0 >> 0$ of the characteristic function χ_Ω for a region Ω. In section 2, we treated the case $\Omega = \Omega_0 \equiv (-\infty, 0] \times \mathbb{R}$, the half-plane. Here, we consider more general Ω obtained by perturbing the half-plane Ω_0.

Condition C. *The sharp confining potential V_Ω is supported in a region Ω so that $\Omega \backslash \Omega_0$ lies in the strip $|y| \leq R < \infty$, for some $0 < R < \infty$.*

We first consider the pair of Hamiltonians (H_0, H_Ω), where $H_0 = H_L(B) + V_0$ is the straight-edge Hamiltonian with sharp confining potential, and $H_\Omega = H_L(B) + V_\Omega$, describes the geometric perturbation of the straight-edge boundary satisfying Condition C. We prove that the local wave operators exist for this pair and that the asymptotic velocity observable is bounded from below by $B^{1/2}$. This observable corresponds to the edge current at $y = \pm\infty$. Furthermore, the spectrum of the perturbed operator H_Ω still has absolutely continuous spectrum between the Landau levels. We then show that this lower bound on the asymptotic velocity observable is stable under a perturbation V_1 that is small compared to the field strength B.

Theorem 4.2 *Let $H \equiv H_L(B) + V_\Omega + V_1$ be the perturbed Hamiltonian with sharp confining potential V_Ω and a bounded perturbation $V_1 \in L^\infty(\mathbb{R}^2)$. Suppose the region $\Omega \backslash \Omega_0$ satisfies Condition C. Let Δ_n be as in (2.6). Let $V_y^\pm(\Delta_n)$ be the asymptotic velocity for the pair (H_0, H_Ω). Suppose that $(c-a)$ and $\|V_1\|_\infty/B$ are sufficiently small as in Theorem 2.3. For any state $\psi = E(\Delta_n)\psi$, the asymptotic edge-current velocity $V_y^\pm(\Delta_n)$ satisfies*

$$\langle \psi, V_y^\pm(\Delta_n)\psi\rangle \geq C_n B^{1/2} \|\psi\|^2. \tag{4.4}$$

We remark that is is not required that the new region Ω be connected nor that it be bounded in the x-direction. The basic situation that we have in mind, however, is the one for which the new region Ω represents a distortion of the boundary of the half-plane Ω_0. It is interesting to note that the edge current persists for some states even if the boundary extends to $+\infty$ along the x-axis. For example, the right half-plane may actually be disconnected if the perturbation is supported in a cone-type region with vertex at $y = 0$ and $x = +\infty$.

Before we prove Theorem 4.2, we consider the effect of the boundary perturbation with $V_1 = 0$. We define $H_0 = H_L(B) + V_0$ and $H_\Omega = H_L(B) + V_\Omega$, and we denote the corresponding spectral families by $E_0(\cdot)$ and $E_\Omega(\cdot)$, respectively. We first prove the existence of the local wave operators for the pair (H_0, H_Ω) by the method of stationary phase. This proves the existence of absolutely continuous spectrum in intervals between Landau levels. We then use these local wave operators to prove the persistence of edge currents. We consider the perturbation of the confining potential $V_0(x)$ given by

$$V_\Omega(x, y) = \mathcal{V}_0(\chi_{(-\infty,0]}(x) + \chi_{\Omega\backslash\Omega_0}(x, y)) = V_0(x) + \mathcal{V}_0\chi_{\Omega\backslash\Omega_0}(x, y), \tag{4.5}$$

and we will write $\delta V \equiv V_\Omega - V_0$, so that $\delta V = \mathcal{V}_0\chi_{\Omega\backslash\Omega_0}(x, y)$. This perturbation of the confining potential is interpreted as a perturbation of the boundary of the region where the electron can propagate.

Proposition 4.1 *Let Δ_n be as in (2.6) with $(c-a)$ sufficiently small. Then, the local wave operators $\Omega_\pm(\Delta_n)$ for the pair (H_0, H_Ω) exist. As a consequence, operator H_Ω has absolutely continuous spectrum in Δ_n.*

Proof. We use Cook's method and study the local operators defined by

$$\begin{aligned}\Omega(t;\Delta_n) - E_0(\Delta_n) &= e^{itH_\Omega}e^{-iH_0 t}E_0(\Delta_n) - E_0(\Delta_n) \\ &= i\int_0^t e^{isH_\Omega}\delta V e^{-iH_0 s}E_0(\Delta_n)\, ds. \qquad (4.6)\end{aligned}$$

Hence, it suffices to prove that for any smooth vector ψ,

$$\lim_{t_1,t_2\to\infty}\int_{t_1}^{t_2} \delta V e^{-isH_0}E_0(\Delta_n)\psi\, ds = 0. \qquad (4.7)$$

In order to prove (4.7), we use the method of stationary phase. Using the partial Fourier transform in (4.7), we have

$$\begin{aligned}&(\delta V e^{-isH_0}E_0(\Delta_n)\psi)(x,y) \\ &= \sum_{j=0}^{n}\delta V(x,y)\int_{\mathbb{R}} e^{-i\omega_j(k)s+iky}\chi_{\omega_j^{-1}(\Delta_n)}(k)\hat{\psi}(x,k)\, dk. \qquad (4.8)\end{aligned}$$

We define the phase as $\Phi(k,y,s) \equiv ky - \omega_j(k)s$, and note that the derivative is $\partial_k\Phi(k,y,s) = y - \omega_j'(k)s$. Let $\chi_R(y)$ be the characteristic function on the interval $[-R,R]$. We have the following lower bound

$$|\partial_k\Phi(k,y,s)\chi_{\omega_j^{-1}(\Delta_n)}(k)\chi_R(y)| \geq |\omega_j'(k)s - y|\chi_{\omega_j^{-1}(\Delta_n)}(k)\chi_R(y). \qquad (4.9)$$

In section 2.2, we proved that there is a constant $C_{n,j} > 0$ such that

$$-\omega_j'(k)\chi_{\omega_j^{-1}(\Delta_n)}(k) \geq C_{n,j}B\chi_{\omega_j^{-1}(\Delta_n)}(k). \qquad (4.10)$$

Using this lower bound (4.10) in the lower bound (4.9), we obtain

$$|\partial_k\Phi(k,y,s)\chi_{\omega_j^{-1}(\Delta_n)}(k)\chi_R(y)| \geq (C_{n,j}Bs - R)\chi_{\omega_j^{-1}(\Delta_n)}(k)\chi_R(y). \qquad (4.11)$$

As a consequence, we can differentiate the phase factor in (4.8) and bound the integral there by

$$\sum_{j=0}^{n}\frac{1}{\langle s\rangle^N}\left|\int_{\omega_j^{-1}(\Delta_n)}(\partial_k^N e^{i\Phi(k,y,s)})\hat{\psi}(x,k)\, dk\right|, \qquad (4.12)$$

where $\langle s\rangle \equiv (1+|s|^2)^{1/2}$. The convergence of the integral in (4.7) follows from this decay and integration by parts using the smoothness of ψ. $\square$

Proposition 4.2 *Assume the hypotheses of Proposition 4.1. For any* $\psi \in E_\Omega(\Delta_n)L^2(\mathbb{R}^2)$*, we have*

$$\langle \psi, V_y^{\pm}(\Delta_n)\psi\rangle \geq C_n B^{1/2}\|\psi\|^2, \tag{4.13}$$

where the constant C_n *is as in Theorem 2.1. That is, the asymptotic velocity* $V_y^{\pm}(\Delta_n)$ *of the edge current carried by the state* $\psi = E_\Omega(\Delta_n)\psi$*, for the perturbed region, is bounded from below by* $B^{1/2}$*.*

Proof. As a consequence of the existence of the wave operators, we have the local intertwining relation

$$\Omega_\pm(\Delta_n)^* E_\Omega(\Delta_n)\psi = E_0(\Delta_n)\Omega_\pm(\Delta_n)^*\psi. \tag{4.14}$$

This intertwining property (4.14) and the definition (4.2) show that

$$\begin{aligned} \langle \psi, V_y^{\pm}(\Delta_n)\psi\rangle &= \langle \psi, \Omega_\pm(\Delta_n)E_0(\Delta_n)V_y E_0(\Delta_n)\Omega_\pm^*(\Delta_n)\psi\rangle \\ &= \langle E_0(\Delta_n)\Omega_\pm^*(\Delta_n)\psi, V_y E_0(\Delta_n)\Omega_\pm^*(\Delta_n)\psi\rangle. \end{aligned} \tag{4.15}$$

The lower bound for the right side of (4.15) follows from Theorem 2.1,

$$\begin{aligned} \langle E_0(\Delta_n)\Omega_\pm(\Delta_n)\psi, V_y E_0(\Delta_n)\Omega_\pm(\Delta_n)\psi &\geq C_n B^{1/2}\|E_0(\Delta_n)\Omega_\pm^*(\Delta_n)\psi\|^2 \\ &\geq C_n B^{1/2}\|\Omega_\pm^*(\Delta_n)\psi\|^2. \end{aligned} \tag{4.16}$$

Since the wave operators are partial isometries, we have $\|\psi\| = \|\Omega_\pm^*(\Delta_n)\psi\|$, which, together with (4.16), proves the lower bound in (4.13). □

We now prove the stability of the edge current with respect to a small perturbation $V_1 \in L^\infty(\mathbb{R}^2)$. Although we do not necessarily know the spectral type of the perturbed Hamiltonian in intervals between the Landau levels, the edge current is stable.

Proof of Theorem 4.2. The proof of Theorem 4.2 follows the same lines of the proof of Theorem 2.3. Given ψ as in the theorem, we decompose it according to the spectral projectors for H and a slightly larger interval $\tilde{\Delta}_n$ containing Δ_n. As in (2.50), we write

$$\psi = E_\Omega(\tilde{\Delta}_n)\psi + E_\Omega(\tilde{\Delta}_n^c)\psi \equiv \phi + \xi. \tag{4.17}$$

We then have the decomposition as in (2.51). We bound $\|\xi\|$ as in (2.52), and in order to bound $\|V_y^{\pm}(\Delta_n)\xi\|$, we note that the asymptotic velocity is bounded by definition, $\|V_y^{\pm}(\Delta_n)\| \leq [(2n+c)B]^{1/2}$, as follows from (4.3). Finally, we note that the matrix element for ϕ satisfies

$$\langle \phi, V_y^{\pm}(\Delta_n)\phi \rangle \geq \tilde{C}_n B^{1/2} \|\phi\|^2, \tag{4.18}$$

by Proposition 4.2. A simple calculation as in the proof of Theorem 2.3 allows us to obtain the lower bound

$$\|\phi\|^2 \geq \left[1 - \left(\frac{c-a}{\tilde{c}-\tilde{a}} + \frac{2\|V_1\|}{B(\tilde{c}-\tilde{a})} \right) \right] \|\psi\|^2, \tag{4.19}$$

so by taking $c - a$ and $\|V_1\|/B$ sufficiently small, we obtain the result (4.4). □

5 One-Edge Geometries and the Spectral Properties of $H = H_0 + V_1$

The unperturbed operator $H_0 = H_L(B) + V_0$ has purely absolutely spectrum and $\sigma(H_0) = [B, \infty)$. In the paper [21], DeBièvre and Pulé proved that perturbations V_1, as in Theorem 2.3, preserve the absolutely continuous spectrum in an interval Δ_n, provided $|\Delta_n| = c - a$ is sufficiently small. We mention this result here for completeness, and for comparison with the situation for two-edge geometries where we will use commutator methods. For a review of commutator methods, we refer the reader to [30, 31, 32]. The proof in [21] relies on the commutator identity

$$i[H_0, y] = 2V_y. \tag{5.1}$$

This commutator shows that an estimate on the edge current is equivalent to an estimate on the positivity of the commutator. This, in turn, provides an estimate on the spectral type of H_0. As we will see, this equivalence, that an estimate on the edge current implies a commutator estimate, no longer holds for two-edge and other, more complicated geometries. This is one of the reasons we presented a different approach to the one-edge geometries in the previous sections.

Continuing with the perturbation theory of H_0, the commutator on the left in (5.1) is invariant under any perturbation of H_0 by a real-valued potential provided V_1 and y have a common, dense domain. It follows immediately from the commutator

$$i[H_0 + V_1, y] = 2V_y, \tag{5.2}$$

and the techniques of Theorem 2.3, that if $c - a$ is small enough, there exists a finite constant $K_n > 0$ such that

$$E(\Delta_n)(i[H, y])E(\Delta_n) \geq K_n E(\Delta_n). \tag{5.3}$$

Since the double commutator is $[[H, y], y] = -2i$, the following theorem now follows from standard Mourre theory (cf. [30]).

Theorem 5.1 *Let V_1 satisfy the conditions of Theorem 2.3. If $c - a$ and $\|V_1\|_\infty / B$ satisfy the smallness conditions of Theorem 2.3 with respect to n and B, then the operator $H = H_0 + V_1$ has only absolutely continuous spectrum on Δ_n.*

Thus, in the half-plane case, the existence of edge currents for each $\psi \in E(\Delta_n)L^2(\mathbb{R}^2)$ is equivalent to the existence of absolutely continuous spectrum. This need not be the case, however, for more complicated edge geometries. For those situations, there may be edge currents carried by states ψ but the spectrum need not be absolutely continuous (cf. [9, 33, 34, 35, 36]).

6 One-Edge Geometries and General Confining Potentials

The analysis used in section 2 can be extended to the case of more general confining potentials with a straight edge. These potentials are described as *soft* potentials, as opposed to the *hard* potentials such as the Sharp Confining Potential or Dirichlet boundary conditions. In general, the soft confining potential V_0, supported on $x \leq 0$, should be rapidly increasing for $x < 0$. There are two classes of soft confining potentials that we can treat: 1) *globally convex potentials*, such as monomials

$$V_0(x) = \mathcal{V}_0 \left(B^{1/2}|x|\right)^p \chi_{(-\infty,0)}(x), \text{ for } p \geq 1, \tag{6.1}$$

and *convex-concave potentials* that are initially convex and then become asymptotically flat, such as

$$V_0(x) = \mathcal{V}_0 \tanh(B^{1/2}|x|)\chi_{(-\infty,0)}(x). \tag{6.2}$$

These two classes of soft confining potentials require that $\mathcal{V}_0$ be sufficiently large, depending on n, where n is the energy level one is studying. For the sake of simplicity we shall restrict ourselves to the potentials given by (6.1) or (6.2), though the results of Theorems 6.1-6.2 can be generalized to a wider class of confining potentials.

We consider the interval Δ_n defined by (2.6). For the unperturbed model $H_0 = H_L + V_0$, we have the following result.

Theorem 6.1 *Let V_0 be the globally convex (resp. convex-concave) confining potential defined by (6.1) (resp. (6.2)) with*

$$\mathcal{V}_0 > (2n+c)^{2p+1} B / \left(\sqrt{\pi} \left(\frac{(a-1)(c-3)}{4(n+1)\mathcal{H}^{(n)}} \right)^2 \right)^p, \tag{6.3}$$

$$(\textit{resp. } \mathcal{V}_0 > (2n+c)/\tanh\left(\sqrt{\pi} \left(\frac{(a-1)(3-c)}{4(n+1)\mathcal{H}^{(n)}(2n+c)} \right)^2 \right)) \tag{6.4}$$

where the constant $\mathcal{H}^{(n)}$ is defined in (2.39). Then, for any $\psi = E_0(\Delta_n)\psi$ having an expansion as in (2.3) with coefficients $\beta_j(k)$, there is a constant $C_n > 0$ so that for all $|\Delta_n|/B$ small enough, we have

$$-\langle \psi, V_y \psi \rangle \geq C_n (a-1)^2 (3-c)^2 \left(\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} dk |\beta_j(k)|^2 \left(\frac{\tilde{V}_j(k)}{V_j(k)^2} \right) \right) B^{1/2}, \tag{6.5}$$

where $V_j(k)$ and $\tilde{V}_j(k)$ are defined by (6.20)-(6.23).

Proof. We prove the result for the globally convex potential (6.1), the case of (6.2) being treated in the same way. We assume that the conditions of Lemma 2.1 are satisfied so that the cross-terms vanish. We begin with the formula for the matrix element $\langle \psi, V_y \psi \rangle$ in (6.5) following from the partial Fourier transform,

$$\begin{aligned} -\langle \psi, V_y \psi \rangle &= \frac{-1}{2B} \sum_{j=0}^{n} \int_{-\infty}^{0} dx \int_{\omega_j^{-1}(\Delta_n)} dk |\beta_j(k)|^2 \varphi_j(x;k)^2 V_0'(x) \\ &\geq \frac{-1}{2B} \sum_{j=0}^{n} \int_{-\infty}^{x_*} dx \int_{\omega_j^{-1}(\Delta_n)} dk |\beta_j(k)|^2 \varphi_j(x;k)^2 V_0'(x), \end{aligned} \tag{6.6}$$

for some $x_* < 0$ we will specify below. The strategy is to obtain a lower bound for $|\varphi_j(x_*;k)|$. We first turn to estimating $|\varphi_j(x_*;k)|$. We use the results of Lemma 2.2. We expand the eigenfunctions $\varphi_j(x;k)$ in terms of the harmonic oscillator eigenfunctions $\psi_m(x;k)$ given in (2.26), as in (2.27). We find that

$$\sum_{m=0}^{n} |\alpha_m^{(j)}(k)|^2 \geq \frac{1}{2B(n+1)}(E_{n+1}(B) - \omega_j(k)), \tag{6.7}$$

and, with P_n denoting the projector onto the subspace of $L^2(\mathbb{R})$ spanned by the first n harmonic oscillator eigenfunctions,

$$|\langle \varphi_j(\cdot,k), V_0 P_n \varphi_j(\cdot,k)\rangle| \geq \frac{1}{2B(n+1)}(\omega_j(k) - E_n(B))(E_{n+1}(B) - \omega_j(k)). \tag{6.8}$$

We also need an upper bound on this matrix element (6.8). From the definition of P_n, we obtain

$$|\langle \varphi_j(\cdot,k), V_0 P_n \varphi_j(\cdot,k)\rangle| \leq \sum_{m=0}^{n} |\alpha_m^{(j)}(k)| \{ I_{j,m}(x_*;k) + II_{j,m}(x_*;k)\} \tag{6.9}$$

where the integrals $I_{j,m}$ and $II_{j,m}$ are given by

$$I_{j,m}(x_*;k) \equiv \int_{-\infty}^{x_*} V_0(x)\ |\varphi_j(x;k)|\ |\psi_m(x;k)|\ dx, \tag{6.10}$$

and

$$II_{j,m}(x_*;k) \equiv \int_{x_*}^{0} V_0(x)\ |\varphi_j(x;k)|\ |\psi_m(x;k)|\ dx. \tag{6.11}$$

We turn now to defining x_*. In light of (6.4) we choose $\epsilon > 0$ small enough in such a way that

$$\mathcal{V}_0 > (2n+c+\epsilon)^{2p+1} B \Big/ \left(\sqrt{\pi} \left(\frac{(a-1)(c-3)}{4(n+1)\mathcal{H}^{(n)}} \right)^2 \right)^p. \tag{6.12}$$

From this $\epsilon > 0$ we define $x_* = x_*(\epsilon)$ as the unique negative real number such that $V_0(x_*) = (2n+c+\epsilon)B$:

$$x_* \equiv -B^{-1/2} \left(\frac{(2n+c+\epsilon)B}{\mathcal{V}_0} \right)^{1/p}. \tag{6.13}$$

By combining (6.12) with (6.13) we notice that

$$(-x_*) < \left(\frac{(a-1)(3-c)}{4(n+1)\mathcal{H}^{(n)}(2n+c+\epsilon)} \right)^2 \left(\frac{\pi}{B} \right)^{1/2}, \tag{6.14}$$

and that the right side of (6.14) is $\mathcal{O}(B^{-1/2})$. Having said that, (6.11) can be estimated using the inequalities $0 \le V_0(x) \le (2n+c+\epsilon)B$ for $x_* \le x \le 0$, and the form of the harmonic oscillator wave function (2.26). We get

$$II_{j,m}(x_*;k) \le (2n+c+\epsilon)\frac{B^{5/4}}{\pi^{1/4}}\frac{\mathcal{H}_m}{\sqrt{2^m m!}}\ |x_*|^{1/2},$$

where the constant $\mathcal{H}_m$ is defined by (2.39). This, together with (6.14), entail

$$II_{j,m}(x_*;k) \le \frac{(\omega_j(k)-E_n(B))(E_{n+1}(B)-\omega_j(k))}{4(n+1)B},\ k \in \omega_j^{-1}(\Delta_n). \quad (6.15)$$

The first integral $I_{j,m}$ is estimated as

$$I_{j,m}(x_*;k) \le \left(\frac{B}{\pi}\right)^{1/4}\frac{\mathcal{H}_m}{\sqrt{2^m m!}}\int_{-\infty}^{x_*} V_0(x)|\varphi_j(x;k)|\ dx. \quad (6.16)$$

We return to (6.9). In light of the lower bound on the matrix element given in (6.8) and the upper bounds on the integrals given in (6.15) and (6.16), we solve for the integral in (6.16). Bearing in mind (2.39) the sums over m in (6.15)–(6.16) are bounded by $\mathcal{H}^{(n)}$, $\sum_{m=0}^{n}|\alpha_m^{(j)}(k)|\frac{\mathcal{H}_m}{\sqrt{2^m m!}} \le \mathcal{H}^{(n)}$, so we end up getting

$$\int_{-\infty}^{x_*} V_0(x)|\varphi_j(x;k)|\ dx \quad (6.17)$$

$$\ge \frac{1}{2\mathcal{H}^{(n)}}\left(\frac{(\omega_j(k)-E_n(B))(E_{n+1}(B)-\omega_j(k))}{2B(n+1)}\right)\left(\frac{\pi}{B}\right)^{1/4}. \quad (6.18)$$

Now $|\varphi_j(x_*;k)|$ can be estimated from this bound and the pointwise upper bound on φ_j in the classically forbidden region and proved in Proposition 8.2 of Appendix 2,

$$|\varphi_j(x;k)| \le |\varphi_j(x_*;k)|e^{-\sqrt{\epsilon B}(x_*-x)},\ x \le x_*, \quad (6.19)$$

since the potential $W_j(t;k) \equiv (Bt-k)^2 + V_0(t) - \omega_j(k) \ge \epsilon B$ for any $k \in \omega_j^{-1}(\Delta_n)$. In light of this upper bound, we define a function $V_j(k)$ by

$$V_j(k) \equiv \int_{-\infty}^{x_*} V_0(x)e^{-\sqrt{\epsilon B}(x_*-x)}dx \ge 0. \quad (6.20)$$

We insert (6.19) into the integral in (6.17), rearrange, and obtain

$$\begin{aligned} & |\varphi_j(x_*;k)| \\ \geq\ & \frac{1}{V_j(k)}\left(\frac{1}{\mathcal{H}^{(n)}}\frac{(\omega_j(k)-E_n(B))(E_{n+1}(B)-\omega_j(k))}{4B(n+1)}\left(\frac{\pi}{B}\right)^{1/4}\right). \end{aligned} \tag{6.21}$$

We return to the expression for the matrix element of the edge current (6.6). We use the lower bound on the eigenfunction $\varphi_j(x;k)$ derived in Proposition 8.2 of Appendix 2:

$$|\varphi_j(x;k)| \geq |\varphi_j(x_*;k)| e^{-\int_x^{x_*}\sqrt{S_j(t;k)}dt},\ \forall x \leq x_*, \tag{6.22}$$

where $S_j(t;k) \equiv W_j(t,k) + \int_{-\infty}^t |W_j'(u;k)| e^{-2\sqrt{\epsilon B}(t-u)}du$. We substitute this expression (6.22) into the right side of (6.6). It will be convenient to introduce another constant $\tilde{V}_j(k)$ defined by

$$\tilde{V}_j(k) \equiv -\int_{-\infty}^{x_*} V_0'(x) e^{-2\int_x^{x_*}\sqrt{S_j(t;k)}dt}\, dx \geq 0. \tag{6.23}$$

Notice that both integrals $V_j(k)$ and $\tilde{V}_j(k)$ converge. Next, using (6.21), we obtain (6.5) with C_n given by (2.47). □

We now consider the perturbation of H_0 by a bounded potential $V_1(x,y)$. As in Section 2.3, we consider a larger interval $\tilde{\Delta}_n$ given by (2.43) with the same midpoint as Δ_n, and prove that the edge current survives if $\|V_1\|_\infty$ is sufficiently small relative to B.

Theorem 6.2 *Let V_0 be as in Theorem 6.1. Let $V_1(x,y)$ denote a bounded potential and $E(\Delta_n)$ be the spectral projection for $H = H_0 + V_1$ and the interval Δ_n. Let $\psi \in L^2(\mathbb{R}^2)$ be a state satisfying $\psi = E(\Delta_n)\psi$, and the following condition. Let $\phi \equiv E_0(\tilde{\Delta}_n)\psi$ have an expansion as in (2.3) with coefficients $\beta_j(k)$ satisfying*

$$\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} |\beta_j(k)|^2 \left(\frac{\tilde{V}_j(k)}{V_j^2(k)}\right) dk \geq (1/2)\|\phi\|^2, \tag{6.24}$$

where $V_j(k)$ and $\tilde{V}_j(k)$ are defined by (6.20)-(6.23). Then, we have,

$$-\langle\psi, V_y\psi\rangle \geq B^{1/2}\ \left((C_n/2)(3-\tilde{c})^2(\tilde{a}-1)^2 - F(n, \|V_1\|/B)\right)\ \|\psi\|^2, \tag{6.25}$$

where C_n is defined in (2.47) and

$$
\begin{aligned}
F(n, \|V_1\|/B) &= \left(\frac{2}{(\tilde{c}-\tilde{a})}\right)^{1/2} \left(\frac{(c-a)}{2} + \frac{\|V_1\|}{B}\right)^{1/2} \left(2n + c + \frac{\|V_1\|}{B}\right)^{1/2} \\
&\times \left[2 + \left(\frac{2}{(\tilde{c}-\tilde{a})}\right)\left(\frac{(c-a)}{2} + \frac{\|V_1\|}{B}\right)\right] \\
&+ \frac{C_n}{2}\left(\frac{2}{(\tilde{c}-\tilde{a})}\right)^2 \left(\frac{(c-a)}{2} + \frac{\|V_1\|}{B}\right)^2 (3-\tilde{c})^2(\tilde{a}-1)^2.
\end{aligned}
$$

Proof. As in the Proof of Theorem 2.3, we first decompose the function ψ as in (2.50) and expand $\langle \psi, V_y \psi \rangle$ as in (2.51). Next, we use (2.54) and (2.55) to bound $|2Re\langle \phi, V_y \xi \rangle| + \langle \xi, V_y \xi \rangle|$, and deduce from Theorem 6.1 and (6.24) that

$$
\begin{aligned}
-\langle \phi, V_y \phi \rangle &\geq C_n (a-1)^2 (3-c)^2 \left(\sum_{j=0}^{n} \int_{\omega_j^{-1}(\Delta_n)} dk |\beta_j(k)|^2 \left(\frac{\tilde{V}_j(k)}{V_j(k)^2} \right) \right) B^{1/2} \\
&\geq (C_n/2)(a-1)^2(3-c)^2 B^{1/2} \|\phi\|^2.
\end{aligned}
$$

Now, by inserting (2.52) in the identity $\|\phi\|^2 = \|\psi\|^2 - \|\xi\|^2$ we get that

$$
\|\phi\|^2 \geq \left(1 - \left(\frac{2}{(\tilde{c}-\tilde{a})}\right)^2 \left(\frac{(c-a)}{2} + \frac{\|V_1\|}{B}\right)^2\right) \|\psi\|^2, \tag{6.26}
$$

so the result follows by elementary computations. □

7 Appendix 1: Basic Properties of Eigenfunctions and Eigenvalues of $h_0(k)$

After reducing the operator $H_0 = -\Delta + V_0$ to the operator $h_0(k)$ on $L^2(\mathbb{R})$ due to the y-translational invariance, we are concerned with studying the properties of $h_0(k)$ defined by

$$
h_0(k) = p_x^2 + (Bx - k)^2 + V_0(x) = p_x^2 + V(x;k), \tag{7.1}
$$

where $p_x^2 = -d^2/dx^2$, and the nonnegative potential $V_0(x) \in L^2_{loc}(\mathbb{R})$. The resolvent of the operator $h_0(k) = p_x^2 + V(x;k)$ is compact since the effective potential $V(x;k) = (Bx - k)^2 + V_0(x)$ is unbounded as $|x| \to \infty$, so the spectrum is discrete with only ∞ as an accumulation point. We denote the eigenvalues of $h_0(k)$ in increasing order and denote them by $\omega_j(k)$, $j \geq 0$. The normalized eigenfunction associated to $\omega_j(k)$ is $\varphi_j(x;k)$. The variational method shows that the domain of $h_0(k)$ is

$$\mathrm{dom}(h_0(k)) = \{\psi \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}; w(x;k)dx),\ (p_x^2 + V(.;k))\psi \in L^2(\mathbb{R})\}, \tag{7.2}$$

with $w(x;k) = (1 + V(x;k))^{1/2}$. It is a subset of $H^2_{loc}(\mathbb{R})$ since the effective potential $V(.;k) \in L^2_{loc}(\mathbb{R})$. We first discuss the regularity properties of the eigenfunctions.

Proposition 7.1 *The eigenfunctions of $h_0(k)$, given by $\varphi_j(.;k)$, are continuously differentiable in $\mathbb{R}$ for any $j \in \mathbb{N}$ and $k \in \mathbb{R}$. Furthermore, an eigenfunction $\varphi_j(.;k) \in C^{n+2}(I)$ for any open subinterval I of $\mathbb{R}$ such that $V_0 \in C^n(I)$, $n \geq 0$.*

Proof. The proof of this proposition follows from the Sobolev Embedding Theorem which gives $H^2_{loc}(\mathbb{R}) \subset C^1(\mathbb{R})$, and the fact that the Schrödinger equation

$$\varphi_j''(x;k) = (V(x;k) - \omega_j(k))\varphi_j(x;k),$$

shows that $\varphi_j''(x;k) \in L^2_{loc}(\mathbb{R})$. $\square$

In the particular case of the Sharp Confining Potential $V_0(x) = \mathcal{V}_0\chi_{(-\infty,0)}(x)$, Proposition 7.1 shows that $\varphi_j(.;k) \in C^1(\mathbb{R}) \cap C^\infty(\mathbb{R}\backslash\{0\})$. Notice that $\varphi_j(.;k)$ is continuously differentiable at the origin although V_0 is discontinuous at this point. For the Parabolic Confining Potential $V_0(x) = \mathcal{V}_0 x^2\chi_{(-\infty,0)}(x)$, we have $\varphi_j(.;k) \in C^3(\mathbb{R}) \cap C^\infty(\mathbb{R}^*)$ since V_0 is only C^1 in any neighborhood of the origin.

We next turn to a proof of the simplicity of the eigenvalues of $h_0(k)$. We state Lemma 7.1 without proof. It is a simple consequence of the Unique Continuation Theorem for Schrödinger Operators (Theorem XIII.63 of [37]). We will use this lemma in the proof of Propositions 7.2 and 8.1.

Lemma 7.1 *Let I be an open (not necessarily bounded) subinterval of $\mathbb{R}$, $W \in L^2_{loc}(I)$ and $\psi \in H^2_{loc}(I)$ satisfy*

$$\psi''(x) = W(x)\psi(x),\ \text{a.e. } x \in I.$$

Then, if ψ vanishes in the neighborhood of a single point $x_0 \in I$, ψ is identically zero in I.

Proposition 7.2 *The eigenvalues $\omega_j(k)$ of the operator $h_0(k)$ are simple for all $k \in \mathbb{R}$.*

Proof. We consider two L^2-eigenfunctions φ and ψ of $h_0(k)$ with same energy E. As follows from Proposition 7.1, they are both $H^2_{loc}(\mathbb{R})$-solutions of the Schrödinger equation

$$u''(x) = (V(x,k) - E)u(x), \ a.e. \ x \in \mathbb{R}. \tag{7.3}$$

By substituting φ (resp. ψ) for u in (7.3), multiplying by ψ (resp. φ), and taking the difference of the two equalities, we get

$$\varphi''(x)\psi(x) - \varphi(x)\psi''(x) = (\varphi'\psi - \varphi\psi')'(x) = 0, \ a.e. \ x \in \mathbb{R}.$$

Consequently, the function $(\varphi'\psi - \varphi\psi')$ is a constant for a.e. x in $\mathbb{R}$, and this constant is zero since the function is in $L^2(\mathbb{R})$ as follows from Proposition 7.1,

$$(\varphi'\psi - \varphi\psi')(x) = 0, \ \forall x \in \mathbb{R}. \tag{7.4}$$

We notice there is always a real number a such that the potential $V(x;k) - E > 0$ for a.e. $x > a$ (since $V(x;k) \to \infty$ as $x \to \infty$) and $\psi(a) \neq 0$ (ψ would be identically zero in $\mathbb{R}$ by Lemma 7.1 otherwise) so $\psi(x) \neq 0$ for any $x > a$ by part 1 of Proposition 8.1. Hence (7.4) implies

$$(\varphi/\psi)'(x) = 0, \ \forall x > a,$$

so we have $\varphi = \lambda\psi$ on $(a, +\infty)$ for some constant $\lambda \in \mathbb{R}$. The function $\varphi - \lambda\psi$ is also an $H^2_{loc}(\mathbb{R})$-solution to (7.3) which vanishes in $(a, +\infty)$. It is also identically zero in $\mathbb{R}$ by Lemma 7.1 hence $\{\varphi, \psi\}$ is a one dimensional manifold of $L^2(\mathbb{R})$. □

8 Appendix 2: Pointwise Upper and Lower Exponential Bounds on Solutions to Certain ODEs

We obtain pointwise, exponential, upper and lower bounds on solutions to the ordinary differential equation $\psi'' = W\psi$, with $W > 0$. We apply these results in the next section to the eigenfunctions $\varphi_j(.;k)$ of $h_0(k)$ in the classically forbidden region where $W_j(x;k) \equiv V(x;k) - \omega_j(k) > 0$. We consider the following general situation. We let ψ denote a *real* $H^1((-\infty,a))$-solution to the system

$$\begin{cases} \psi''(x) = W(x)\psi(x), \; a.e. \; x < a \\ \lim\limits_{x \to a^-} \psi(x) = \psi(a) > 0, \end{cases} \tag{8.1}$$

for some $a \in \mathbb{R}$, where $W \in L^2_{loc}((-\infty,a))$ is such that :

$$W(x) > 0, \; a.e. \; x < a. \tag{8.2}$$

Standard arguments already used in the proof of Proposition 7.1, assure us that the solution $\psi \in H^2_{loc}((-\infty,a))$ so $\psi \in C^1((-\infty,a))$. Moreover ψ is left continuous at a, according to (8.1).

8.1 Basic Properties of ψ

We prove the following basic result that characterizes the behavior of the solution ψ in the classically forbidden region where $W(x) > 0$.

Proposition 8.1 *Any real $H^1((-\infty,a))$-solution ψ to (8.1) satisfies :*

1. *$\psi(x) > 0$ and $\psi'(x) > 0$, for any $x < a$;*
2. *$\lim\limits_{x \to -\infty} W(x)\psi^2(x) = 0$.*

We prove the first part of Proposition 8.1 in two elementary lemmas.

Lemma 8.1 *Under the hypotheses of Proposition 8.1, suppose that $\psi(x_0)\psi'(x_0) < 0$, for some $x_0 < a$. If $\psi(x_0) > 0$, we have $\psi(x) > \psi(x_0)$, for any $x < x_0$, and if $\psi(x_0) < 0$, we have $\psi(x_0) > \psi(x)$, for any $x < x_0$. Consequently, we have $\psi(x)\psi'(x) \geq 0$, for any $x < a$.*

Proof. We assume that $\psi(x_0) > 0$ so that the hypothesis implies that $\psi'(x_0) < 0$. The case $\psi(x_0) < 0$, implying $\psi'(x_0) > 0$, is treated in the same manner. Notice that $\mathcal{E} = \{\delta > 0 \mid \psi(x) > \psi(x_0), \text{for } x \in (x_0 - \delta, x_0)\} \neq \emptyset$, since $\psi'(x_0) < 0$, so

$$\delta_0 = \sup \mathcal{E} > 0.$$

If $\delta_0 < \infty$, then $x_1 = x_0 - \delta_0$ satisfies

$$\begin{cases} \psi(x) > \psi(x_0) \ \forall x \in (x_1, x_0) \\ \psi(x_1) = \psi(x_0). \end{cases}$$

Thus for a.e. $x \in [x_1, x_0)$, we have $\psi''(x) = W(x)\psi(x) \geq W(x)\psi(x_0) > 0$ hence $\psi'(x) < \psi'(x_0) < 0$ for all $x \in [x_1, x_0)$, so we finally get

$$\psi(x_1) > \psi(x_0).$$

Actually $\psi(x_1) = \psi(x_0)$, hence $\delta_0 = +\infty$ and the first result follows. Finally, if there is some $x_0 < a$ such that $\psi(x_0)\psi'(x_0) < 0$, then the first result implies that $|\psi(x)| \geq |\psi(x_0)| > 0$, for any $x \leq x_0$. This is impossible since $\psi \in L^2((-\infty, a))$. □

We next consider the possibility that the wave function has zeros in the classically forbidden region.

Lemma 8.2 *Under the hypotheses of Proposition 8.1, we have $\psi(x) > 0$ for any $x < a$.*

Proof.
1. We first show that $\psi(x)\psi'(x) > 0$, for any $x < a$ such that $\psi(x) \neq 0$. We assume that $\psi(x) > 0$ (the case $\psi(x) < 0$ being treated in the same way) so $\psi(t) > 0$ for any $t \in (x - \delta, x)$ for some $\delta > 0$ and $\psi''(t) = W(t)\psi(t) > 0$ for a.e. t in $(x - \delta, x)$. If $\psi'(x) = 0$ we have $\psi'(t) < 0$ and also $\psi(t)\psi'(t) < 0$ for each $t \in (x - \delta, x)$. This is impossible according to Lemma 8.1. Hence $\psi'(x) > 0$ since $\psi'(x) \geq 0$ by Lemma 8.1.
2. Next we show that if $\psi(x_0) = 0$, for some $x_0 < a$, then $\psi'(x_0) = 0$. We assume that $\psi(x_0) = 0$ and $\psi'(x_0) > 0$ (the case $\psi'(x_0) < 0$ being treated in the same manner). In this case we can find some $\delta > 0$ such that $\psi(x) < 0$ and $\psi'(x) > 0$, for any $x \in (x_0 - \delta, x_0)$, which is impossible according to Lemma 8.1.
3. To complete the proof, we assume that there is a real number $x_0 < a$ such that $\psi(x_0) = 0$. We also have $\psi'(x_0) = 0$ by part 2 and

$$\sup\{x < x_0 \mid \psi(x) \neq 0\} = x_0,$$

since ψ would be zero on $(-\infty, a)$ otherwise by Lemma 7.1. Thus, we can find some $\delta > 0$ such that $\pm\psi(x) > 0$, for all $x \in (x_0 - \delta, x_0)$, so $\pm\psi''(x) =$

$W(x)(\pm\psi(x)) > 0$ a.e. in $(x_0 - \delta, x_0)$. This implies that $\pm\psi'(x) < 0$, and, consequently, that $\psi(x)\psi'(x) < 0$, for any $x \in (x_0 - \delta, x_0)$. This is impossible according to Lemma 8.1. □

To justify the second part of Proposition 8.1, we multiply (8.1) by ψ, and integrate over $[x, x_0]$, for some $x_0 < a$ and $x < x_0$. We obtain

$$\int_x^{x_0} \psi''(u)\psi(u)du = \int_x^{x_0} W(u)\psi^2(u)du. \tag{8.3}$$

Integrating by parts in the left side of (8.3), we get

$$\psi(x_0)\psi'(x_0) - \psi(x)\psi'(x) - \int_x^{x_0} \psi'^2(u)du = \int_x^{x_0} W(u)\psi^2(u)du, \tag{8.4}$$

so by taking the limit $x \to -\infty$ in (8.4), we obtain the inequality :

$$0 \leq \int_{-\infty}^{x_0} W(u)\psi^2(u)du \leq \psi(x_0)\psi'(x_0) - \int_{-\infty}^{x_0} \psi'^2(u)du < \infty.$$

Hence, the function $W\psi^2 \in L^1((-\infty, x_0))$, and the result follows.

8.2 Pointwise Bounds

We first compute an upper bound to an $H^1((-\infty, a))$-solution to (8.1) for a potential W bounded from below.

Lemma 8.3 *If $W \in L^2_{loc}((-\infty, a))$ is bounded from below,*

$$W(x) \geq W_m > 0, \text{ a.e. } x < a, \tag{8.5}$$

then any real $H^1((-\infty, a))$-solution of (8.1) satisfies :

$$\psi(x) \leq \psi(x_0)\mathrm{e}^{-W_m^{1/2}(x_0 - x)}, \ \forall x \leq x_0 \leq a. \tag{8.6}$$

Proof. We multiply (8.1) by $\psi'(u)$ so we get

$$\psi''(u)\psi'(u) = W(u)\psi(u)\psi'(u) \geq W_m\psi(u)\psi'(u), \text{ a.e. } u < a,$$

according to (8.5) and Part 1 of Proposition 8.1. Next we integrate this inequality over $[x,t]$ for $x<t<a$, get $\psi'^2(t)-\psi'^2(x)\geq W_m(\psi^2(t)-\psi^2(x))$, and take the limit as $x\to-\infty$:

$$\psi'^2(t)\geq W_m\psi^2(t),\ \forall t<a.$$

This leads to $\psi'(t)\geq W_m^{1/2}\psi(t)$ for any $t<a$, by part 1 of Proposition 8.1. By integrating over $[x,x_0]$, $x\leq x_0<a$, we finally obtain

$$\psi(x)\leq\psi(x_0)\mathrm{e}^{-W_m^{1/2}(x_0-x)}.$$

This result continues to hold for $x_0=a$ since ψ is left continuous at a. □

We then examine the behavior of an $H^1((-\infty,a))$-solution to (8.1) for a potential

$$W\in H^1_{loc}((-\infty,a)). \tag{8.7}$$

The main result on L^2-solutions of the equation (8.1) is the following:

Proposition 8.2 *Let W satisfy (8.5)-(8.7) together with the condition:*

$$\int_{-\infty}^{a}|W'(u)|\mathrm{e}^{2W_m^{1/2}u}du<\infty. \tag{8.8}$$

Then any real $H^1((-\infty,a))$-solution ψ to (8.1) satisfies

$$\psi(x_0)\mathrm{e}^{-\int_x^{x_0}\sqrt{S(t)}dt}\leq\psi(x)\leq\psi(x_0)\mathrm{e}^{-W_m^{1/2}(x_0-x)},\ \textit{for } x\leq x_0\leq a,$$

where $S(t)=W(t)+\int_{-\infty}^{t}|W'(u)|\mathrm{e}^{-2W_m^{1/2}(t-u)}du$, for all $t\leq a$.

Proof.
The left inequality being already given by Lemma 8.3 we only need to prove right one. To do that we multiply (8.1) by $\psi'(x)$ and integrate over $[u,t]$, for $u<t<a$:

$$\int_u^t\psi'(x)\psi''(x)dx=\frac{\psi'^2(t)-\psi'^2(u)}{2}=\int_u^t W(x)\psi(x)\psi'(x)dx.$$

Next, integrating by parts, the right side of this equality gives

$$\psi'^2(t)-\psi'^2(u)=W(t)\psi^2(t)-W(u)\psi^2(u)-\int_u^t W'(x)\psi^2(x)dx,$$

the above integral being well defined since $W' \in L^2_{loc}((-\infty, a))$ and ψ is bounded in $[u, t]$. Taking the limit as $u \to -\infty$ in the previous equality leads to

$$\psi'^2(t) = W(t)\psi^2(t) - \int_{-\infty}^{t} W'(u)\psi^2(u)du, \ \forall t < a, \tag{8.9}$$

according to part 2 of Proposition 8.1. Now we insert the inequality (8.6) written for $u < t < a$

$$\psi(u) \leq \psi(t)\mathrm{e}^{-W_m^{1/2}(t-u)dv},$$

into the following obvious consequence of (8.9)-(8.8):

$$\psi'^2(t) \leq W(t)\psi^2(t) + \int_{-\infty}^{t} |W'(u)|\psi^2(u)du, \ \forall t < a,$$

getting

$$\psi'^2(t) \leq S(t)\psi^2(t), \ t < a.$$

Thus $\psi'(t) \leq \sqrt{S(t)}\psi(t)$ for all $t < a$, by part 1 of Proposition 8.1, so we get

$$\psi(x) \geq \psi(x_0)\mathrm{e}^{-\int_x^{x_0}\sqrt{S(t)}dt}, \ \forall x \leq x_0 < a, \tag{8.10}$$

by integrating over $[x, x_0]$. Taking account of the left continuity of ψ at a we extend this result at $x_0 = a$ by taking the limit in (8.10) as $x_0 \to a$. □

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