Dynamical localization in periodically driven quantum systems

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Abstract. Given a periodically time dependent Hamiltonian $H(t)$ and $\psi(t)$ the solution at time t of the Schrödinger equation $(-i\partial_t + H(t))\psi(t) = 0$, we shortly review what is known on the time behaviour of $E_{\psi(0)}(t) := (H(t)\psi(t), \psi(t))$ as t tends to infinity. We give sufficient conditions to insure that $E_{\psi(0)}$ remains bounded in the course of time, a phenomenon that we call *dynamical localization* of the quantum trajectory, $\{\psi(t),\, t \in \mathbb{R}\}$. We also deduce bound, uniform in time, on transition probabilities for such systems.

1 Introduction

1.1 What we are aiming for

Let $\mathbb{R} \ni t \mapsto H(t)$ be time dependent quantum Hamiltonian acting on a separable Hilbert space $\mathcal{H}$ which is T-periodic, $T > 0$. Assume that $H(t)$ is selfadjoint for all t and that the Schrödinger equation

$$-i\partial_t\psi(t) + H(t)\psi(t) = 0, \quad \text{with initial condition} \quad \psi(0) \in \mathcal{H} \tag{1}$$

has a unique solution $\mathbb{R} \ni t \mapsto \psi(t) \in \operatorname{dom} H(t)$, i.e. $\psi(t)$ belongs to the domain of $H(t)$. Let

$$E_{\psi(0)}(t) := (H(t)\psi(t), \psi(t)).$$

The question we want to address in this article is *what are sufficient conditions on $H(t)$ and $\psi(0)$ so that*

$$\sup_{t\in\mathbb{R}} \left| E_{\psi(0)}(t) \right| < \infty? \tag{2}$$

This question may be rephrased in physical terms as *whether the quantum system described by $H(t)$ cannot pump energy indefinitely from outside?* Indeed since the Hamiltonian plays the role of the energy observable in quantum mechanics, $E_{\psi(0)}(t)$ is nothing but the energy of the system in the quantum state $\psi(t)$. If $H(t)$ would be time independent, then $t \mapsto E_{\psi(0)}(t)$ is obviously constant. Time dependent Hamiltonian are designed to model the action of an external force on the quantum system under study and (2) is the question whether this force leaves the energy of the system bounded in the course of time .

We shall consider only quantum systems which have a purely punctual Floquet operator $U(T,0) : \mathcal{H} \mapsto \mathcal{H}$, where $U(t,s)$ denotes the propagator associated to $H(t)$, i.e. the operator which transforms $\psi(s)$ into $\psi(t)$.

Another restriction we make in these notes is

$$V(t) := H(t) - H(0) \quad \text{is strongly } C^1(\mathbb{R}, \mathcal{B}(\mathcal{H})) \tag{3}$$

so that (2) will be true if and only if $(H(0)\psi(t), \psi(t))$ remains bounded. Let $P(\Delta)$ denote the spectral projector of $H(0)$ associated to the real interval Δ and assume that $\psi(0) \in \operatorname{Ran} P(\Delta)$. Such a property (2) for $H(0)$ requires that

$$\lim_{\operatorname{dist}(\Delta',\Delta)\to\infty} \sup_{t\in\mathbb{R}} \|P(\Delta')U(t,0)P(\Delta)\|^2 = 0 \tag{4}$$

i.e. the *transition probability* $\|P(\Delta')U(t,0)P(\Delta)\|^2$ to jump in time t from a given spectral subspace of $H(0)$ to a high energy one, $\operatorname{Ran} P(\Delta')$, must vanish uniformly in time as $\inf \Delta' \to \infty$. We shall also consider this question below.

In view of the RAGE theorem for periodically driven systems, see [EV] and §1.2 below, it is natural to demand that $U(T,0)$ is purely punctual in order to get (2) and (4). However if this guarantees the relative compactness of the quantum trajectories, $\{\psi(t),\, t > 0\}$, it does not seem sufficient to get (2). Even, one knows for stationary pure point Schrödinger Hamiltonians in $L^2(\mathbb{Z}^d)$ or $L^2(\mathbb{R}^d)$, $d \geq 1$, that $t \mapsto (X^2\psi(t), \psi(t))$ has no ballistic behaviour, see [Si],

$$\limsup_{t\to\infty} (X^2\psi(t), \psi(t)) \frac{1}{t^2} = 0$$

but it is shown in [RiJLSi] that a subballistic behaviour may happen

$$\limsup_{t\to\infty} (X^2\psi(t), \psi(t)) \frac{\log t}{t^2} = +\infty.$$

In this context one says that there is dynamical localization if

$$\sup_{t\in\mathbb{R}} (X^2\psi(t)\psi(t)) < \infty.$$

By analogy we shall say that *there is dynamical localization of the trajectory with initial condition $\psi(0)$ in the periodically driven quantum system described by $H(t)$ if (2) is verified.*

1.2 What is known so far

It is classical, see e.g. [Yo] that under (3) the Schrödinger equation (1) possesses a unique solution which is strongly differentiable, for each initial condition in $\mathrm{dom}\, H(t) = \mathrm{dom}\, H(0)$. Under this assumptions (3) it is obvious (since $\partial_t E_{\psi(0)}(t) = (V'(t)\psi(t), \psi(t))$) to get

$$\left| E_{\psi(0)}(t) \right| \leq \int_0^t \|V'(s)\| \|\psi(0)\|^2 ds \leq \sup_{t\in[0,T]} \|V'(t)\| \|\psi(0)\|^2 \, t$$

i.e. the energy can at most grow linearly in time. A weaker growth can be proven for system having increasing spectral gaps. The first result of that type is due to A. Joye [J]. A more general result due to G. Nenciu [N] is as follows:

Theorem N. *Let $H := H_0 + V(t)$ with $H_0 \geq 0$, selfadjoint, V symmetric and $V \in C^m(\mathbb{R}, \mathcal{B}(\mathcal{H}))$ in norm; V is not assumed to be periodic but uniformly bounded. Suppose that the spectrum of H_0 has the following structure: $\exists$ $0 < c \leq C < \infty$, $\alpha > 0$, $d < \infty$*

$$\mathrm{spect}\,(H_0) = \bigcup_{1\leq j} \mathrm{sp}_j, \quad c\, j^\alpha \leq \mathrm{dist}\,(\mathrm{sp}_j, \mathrm{sp}_{j+1}) \leq C j^\alpha, \quad \mathrm{diam}\,(\mathrm{sp}_j) \leq d\, j^\alpha.$$

$$\textit{If } m \geq \left[\tfrac{1+\alpha}{2\alpha}\right] \textit{ then:} \quad \limsup_{t\to\infty} \frac{E_{\psi(0)}(t)}{t^{\frac{1+\alpha}{m\alpha}}} < \infty, \quad \forall \psi(0) \in \mathrm{dom}\, H_0.$$

If in addition V is periodic, small enough with appropriate frequencies one can get uniform boundedness of the energy. The following theorem is proven in [ADE]:

Theorem ADE. *Let $H = H_0 + gV(t)$ with H_0 selfadjoint $g > 0$, V in $C^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$, symmetric and $2\pi/\omega$- periodic. Assume that $\omega \in \Omega$ where Ω is a compact interval which does not contain the origin. Assume also that H_0 has discrete simple spectrum $\{E_n\}_{n\in\mathbb{N}}$ which verifies:*

$$\exists \alpha > 0, \quad \inf_{n\in\mathbb{N}} \frac{E_{n+1} - E_n}{n^\alpha} > 0.$$

Then there exists $g_0 > 0$ so that for all $0 < g \leq g_0$, one can find $\tilde{\Omega} \subset \Omega$ such that:

(i) $\exists C > 0$, $|\Omega \setminus \tilde{\Omega}| \leq C\sqrt{g}$,

(ii) $\forall \omega \in \tilde{\Omega}, \forall \psi(0) \in \operatorname{dom} H_0, \quad \sup_t E_{\psi(0)}(t) < \infty$.

We end up this short review of results about the question (2) by mentioning the article [DeBF] where an interesting panorama around this question is proposed.

Concerning the question (4) about smallness of transitions probabilities uniform in time, there is the following byproduct of the RAGE theory for periodically driven systems, see [EV]:

Theorem EV. *If $U(T,0)$ is pure point, then every quantum trajectory has a compact closure and for any family of bounded operators $\{P_s\}_{s\in\mathbb{R}_+}$ which converges strongly to the zero as $s \to \infty$ one has*

$$\lim_{s\to\infty} \sup_{t>0} \|P_s U(t,0)\psi(0)\| = 0.$$

The aim of this article is to extend theorems ADE and EV.

2 Floquet decomposition revisited

2.1 Floquet operator and Floquet Hamiltonian

Assume we are given on a separable Hilbert space $\mathcal{H}$ a strongly continuous and unitary propagator $\mathbb{R} \times \mathbb{R} \ni (t,s) \mapsto U(t,s)$ which is T-periodic, i.e.:

$$\exists T > 0, \quad \forall t,s \in \mathbb{R}, \quad U(t+T, s+T) = U(t,s). \tag{5}$$

Let

$$\mathcal{K} := L^2(\omega^{-1}S^1, \mathcal{H}) \sim L^2(\omega^{-1}S^1) \otimes \mathcal{H}, \qquad \omega := \frac{2\pi}{T}$$

be the so-called *extended Hilbert space* of T-periodic and L^2-function with values in $\mathcal{H}$. In this article S^1 denotes the circle of radius 1. We shall follow the Howland-Yajima construction [H1, Ya].

The following mapping induces a one parameter (σ) unitary group on $\mathcal{K}$ which is strongly continuous:

$$\forall \sigma \in \mathbb{R}, \quad (\mathcal{V}_\sigma f)(t) := U(t, t-\sigma) f(t-\sigma). \tag{6}$$

By Stone's theorem we get a selfadjoint operator K in $\mathcal{K}$ so that $\mathcal{V}_\sigma = \exp(-i\sigma K)$ with the following characterization of its domain

$$\operatorname{dom} K \ni f \iff i\partial_\sigma \mathcal{V}_\sigma f|_{\sigma=0} \in \mathcal{K}.$$

It is straightforward to verify that

$$e^{-iTK} = \mathcal{W}\, 1 \otimes U(T,0)\, \mathcal{W}^{-1}, \quad \text{with} \quad \mathcal{W} := \int_{\omega^{-1}S^1}^{\oplus} U(t,T)dt \tag{7}$$

i.e. $\exp(-iTK)$ and $1 \otimes U(T,0)$ are unitarily equivalent.

If in addition we know that U is generated by a selfadjoint family of periodic Hamiltonians $\{H(t)\}$ satisfying (3) then K is the closure of

$$D \otimes 1 + \int_{\omega^{-1}S^1}^{\oplus} H(t)dt, \quad \text{with domain} \quad \mathcal{H}^1(\omega^{-1}S^1) \otimes \operatorname{dom} H(0)$$

where $D := -i\partial_t$ with domain $\mathcal{H}^1(\omega^{-1}S^1)$ (the first Sobolev space). K and $U(T,0)$ are called respectively *Floquet Hamiltonian* and *Floquet operator*.

2.2 Floquet decomposition

Let $M_\omega : \mathcal{K} \mapsto \mathcal{K}$ denote the multiplication operator by $\exp(i\omega t)$. Thanks to (5) one gets

$$M_\omega \operatorname{dom} K = \operatorname{dom} K \quad \text{and} \quad M_\omega K M_\omega^{-1} = K - \omega. \tag{8}$$

Assume now that

$$K \textit{ is purely punctual} \tag{9}$$

i.e. $\mathcal{K}$ possesses a basis of eigenvectors of K. We shall denote by f these eigenvectors and F_f the corresponding eigenvalue: $Kf = F_f f$. Thanks to (8), by Hausdorff's maximality principle there exists an orthonormal eigenbasis $\mathcal{F}$ of K which is M_ω invariant. This allows to define an equivalence relation on $\mathcal{F}$

$$f \sim g \quad \Longleftrightarrow \quad \exists n \in \mathbb{Z},\ f = M_\omega^n g.$$

We call $\mathcal{F}_0$ a subset of $\mathcal{F}$ made of exactly one element in each equivalence class.

Each eigenvector $Kf = F_f f$ defines a strongly continuous function $\omega^{-1}S^1 \ni t \mapsto f(t) \in \mathcal{H}$ since

$$\exists t_0 \in \omega^{-1}S^{,} \quad f(t) = e^{i(t-t_0)F_f} U(t,t_0) f(t_0) \tag{10}$$

which follows from an application of Fubini's theorem to (6) and since U is strongly continuous. Thus we may set

$$\forall t \in \omega^{-1}S^1, \quad \mathcal{F}_0(t) := \{f(t) \in \mathcal{H},\ f \in \mathcal{F}_0\}.$$

It is straightforward to show that

$$\forall t \in \omega^{-1} S^1, \quad \sqrt{T}\mathcal{F}_0(t) \textit{ is an orthonormal basis of } \mathcal{H}.$$

We define now two operators

$$U_A(t,s)f(s) := f(t), \quad H_A f(0) := F_f f(0), \quad \forall f \in \mathcal{F}_0$$

and get by rewriting (10) the well known *Floquet decomposition* of the propagator U:

$$\forall t,s \in \omega^{-1} S^1 \quad U(t,s) = U_A(t,0)e^{-i(t-s)H_A}U_A(0,s). \tag{11}$$

Notice that U_A is unitary and H_A is selfadjoint. In the sequel U and U_A will also denote their extension to $\mathbb{R} \times \mathbb{R}$ by periodicity.

3 A criterion for dynamical localization

We start by an easy to get criterion:

theorem 1. *Assume (3) and that V is T-periodic and let U denote the propagator which solves (1). If $U(T,0)$ is purely punctual and possesses an eigenbasis $\mathcal{B}$ which is contained in $\operatorname{dom} H(0)$ then for all linear and finite combination ψ of elements of $\mathcal{B}$:*

$$\sup_{t \in \mathbb{R}} |E_\psi(t)| < \infty.$$

Proof. Let $U(T,0)\varphi_i = \lambda_i \varphi_i$, $i = 1,2$ then $\varphi_i \in \operatorname{dom} H(0)$ by assumption so that as we recalled above, $t \mapsto \varphi_i(t) := U(t,0)\varphi_i$ is strongly C^1 and therefore $E_{\varphi_1,\varphi_2}(t) := (H(t)\varphi_1(t), \varphi_2(t)) = i(\varphi_1'(t), \varphi_2(t))$ is continuous with respect to t. Next since H and U are T-periodic (see (5)) the same is true for $t \mapsto |E_{\varphi_1,\varphi_2}(t)|$. Indeed

$$\begin{aligned} |E_{\varphi_1,\varphi_2}(t+T)| &= |(H(t+T)U(t+T,T)U(T,0)\varphi_1, U(t+T,T)U(T,0)\varphi_2)| \\ &= |\lambda_1\overline{\lambda_2}(H(t)U(t,0)\varphi_1, U(t,0)\varphi_2)| = |E_{\varphi_1,\varphi_2}(t)| \end{aligned}$$

since $|\lambda_1\overline{\lambda_2}| = 1$. We may conclude that E_{φ_1,φ_2} is uniformly bounded on $\mathbb{R}$. Finally let $\psi := \sum_{i=1}^N c_i\varphi_i$, with $\varphi_i \in \mathcal{B}$ and normalized, and $c_i \in \mathbb{C}$, for all i, an easy estimation gives

$$|E_\psi(t)| \leq \sup_{1 \leq i,j \leq N} |E_{\varphi_i,\varphi_j}(t)| N \|\psi\|^2$$

which shows the desired statement. ∎

Remarks. (a) The above result extends obviously to the case where U is purely punctual only on a subspace of $\mathcal{H}$.

(b) In the conditions of the above theorem there is a dense set of initial conditions with a dynamically localized trajectory. However we do not know if this is true for all initial conditions in $\operatorname{dom} H(0)$. This is the question we shall consider in the remainder of this section.

(c) It can be shown that $U(T,0)$ possesses an eigenbasis in $\operatorname{dom} H(0)$ iff the Floquet Hamiltonian K has an eigenbasis $\mathcal{F}$ such that $\forall f \in \mathcal{F}$, $t \mapsto f(t) \in \mathcal{H}$ is strongly C^1 . This motivates our assumption (DL1) below.

Assumption DL.

$$\textit{For all } f \in \mathcal{F}_0,\ f : \omega^{-1}S^1 \mapsto \mathcal{H} \textit{ is strongly differentiable.} \tag{DL1}$$

If we define $S : \mathcal{K} \mapsto \mathcal{K}$ *by*

$$\forall f \in \mathcal{F}_0,\ \forall t \in \omega^{-1}S^1, \quad S(t)f(t) := -if'(t)$$

then

$$S \in \mathcal{B}(\mathcal{K}). \tag{DL2}$$

Remarks. (a) Assume (3), then $t \mapsto U(t,0)g$ is strongly differentiable for all $g \in \operatorname{dom} H(0)$. It follows with the help of (10) that every eigenvector f of K is weakly differentiable i.e.

$$\forall g \in \operatorname{dom} H(0), \qquad t \mapsto (f(t), g) = e^{itF_f}(f(0), U(t,0)^\star g)$$

is differentiable. However it is not clear at all that f is strongly differentiable. Thus (DL1) seems to be an extra quality imposed on the system; notice that it is equivalent to $f(0) \in \operatorname{dom} H(0)$ for all eigenvector f of K as we announced in the above remark.

(b) It is a simple exercise to verify that $t \mapsto S(t)$ is strongly continuous.

(c) Notice that (DL2) is equivalent to $\sup_t \|S(t)\| < \infty$.

(d) S is selfadjoint and $-S$ is the generator of U_A. Indeed for all eigenvectors f of K

$$-i\partial_t U_A(t,0)f(0) = -if'(t) = S(t)f(t).$$

Also:

$$S(t) = -i(\partial_t U_A(t,0))U_A(t,0)^\star. \tag{12}$$

Let

$$U_A(t) := U_A(t,0) \quad \text{and} \quad U(t) := U(t,0).$$

With this S operator we may rewrite the eigenvalue equation $Kf = F_f f$ as $(S(t) + H(t))f(t) = U_A(t)H_AU_A(t)^{-1}f(t)$ so that by density one gets

$$S + H = U_AH_AU_A^{-1} = UH_AU^{-1} \qquad \text{on dom}\, K;$$

the last equality follows from (11). In particular

$$S(0) + H(0) = H_A \quad \text{and} \quad HU = -SU + UH_A$$

so that for all $\psi \in \text{dom}\, H(0)$:

$$\|HU\psi\| = \|(-SU + U(H(0) + S(0)))\psi\| \leq 2\|S\|\|\psi\| + \|H(0)\psi\|. \tag{13}$$

One also has

$$(S + V)U - US(0) = [U, H(0)]. \tag{14}$$

Let now $P(\Delta)$ and $P(\Delta')$ be two spectral projectors of $H(0)$.

Theorem 2. *Assume (3) and (DL) then*

$$\forall\psi \in \text{dom}\, H(0), \quad \sup_{t\in\mathbb{R}} |E_\psi(t)| < \infty$$

and

$$\sup_{t\in\mathbb{R}} \|P(\Delta)U(t,0)P(\Delta')\| \leq \frac{\pi}{2\text{dist}\,(\Delta,\Delta')}(2\|S\| + \|V\|)$$

Proof. One has $|E_\psi(t)| = |(H(t)U(t)\psi, U(t)\psi)| \leq 2\|S\|\|\psi\|^2 + \|H(0)\psi\|\|\psi\|$ using (13). To prove the second statement we remark that

$$H(0)P(\Delta)P(\Delta)UP(\Delta') - P(\Delta)UP(\Delta')P(\Delta')H(0) = P(\Delta)[H(0),U]P(\Delta')$$

which is of the form $AX - XB = Y$. Such an equation in the unknown X may be solved, see [BhaRos], and

$$\begin{aligned} \|P(\Delta)UP(\Delta')\| &\leq \frac{\pi}{2}\frac{1}{\text{dist}\,(\Delta,\Delta')}\|[H(0), P(\Delta)UP(\Delta')]\| \\ &\leq \frac{\pi}{2}\frac{1}{\text{dist}\,(\Delta,\Delta')}\|[H(0),U]\|. \end{aligned}$$

The rest follows from (14). ∎

The second statement of the above theorem improves the result of the EV theorem, at least for the particular family $\{P_s\}_s$ made of the spectral projections of $H(0)$.

4 Sufficient conditions for KAM systems

Let $K_0 := D \otimes 1 + 1 \otimes H(0)$ acting in $\mathcal{K}$. An orthonormal eigenbasis of $D : L^2(\omega^{-1}S^1) \mapsto L^2(\omega^{-1}S^1)$ is

$$\{\chi_{m_1},\, m_1 \in \mathbb{Z}\}, \quad \text{with} \quad \chi_{m_1}(t) = \frac{1}{\sqrt{T}} e^{im_1\omega t}.$$

We start by proving the

lemma 3. *Assume: (i) there exists a unitary operator $\mathcal{U}$ and a bounded selfadjoint operator G such that $K_0 + V = \mathcal{U}(K_0 + G)\mathcal{U}^\star$, with $[D, G] = [H_0, G] = 0$ and $[\mathcal{U}, e^{it}] = 0$.*
(ii) $K_0 + G$ is pure point.

Then

(a) K_0+G has an orthonormal eigenbasis of the form $\{\chi_{m_1} \otimes \varphi_{m_2}\}_{(m_1,m_2)\in \mathbb{Z}\times\mathbb{N}}$ where $\{\varphi_{m_2}\}_{m_2 \in \mathbb{N}}$ is an orthonormal eigenbasis of $H(0)$.
(b) If $\mathcal{F}_0$ is chosen as $\{f_{0,m_2} := \mathcal{U}\chi_0 \otimes \varphi_{m_2}\}_{m_2}$ it follows that

$$\forall t, \quad S(t) = -i(\partial_t \mathcal{U}(t))\mathcal{U}^\star(t). \tag{15}$$

Proof. (a) Since $[D, G] = 0$ this means that G does not depend on time. Thus $K_0 + G = D \otimes 1 + 1 \otimes (H_0 + G)$. This shows (a).

(b) We shall use the formula (12), and prove that with the above choice of $\mathcal{F}_0$ one has $U_A(t) = \mathcal{U}(t)\mathcal{U}^\star(0)$; then clearly $S(t) = -i(\partial_t \mathcal{U}(t))\mathcal{U}^\star(t)$ follows.

So it remains to verify that $U_A(t) = \mathcal{U}(t)\mathcal{U}^\star(0)$. We recall that $U_A(t)$ is defined by: $U_A(t)f_{0,m_2}(0) = f_{0,m_2}(t)$. By definition of f_{0,m_2} and since $[\mathcal{U}, e^{it}] = 0$ one has

$$\forall t, \quad \mathcal{U}(t)\frac{1}{\sqrt{T}}\varphi_{m_2} = f_{0,m_2}(t)$$

so that in particular: $\mathcal{U}(0)\frac{1}{\sqrt{T}}\varphi_{m_2} = f_{0,m_2}(0)$. Our statement follows at once. ∎

Because of (15), we see that in order to apply the criterion of §3 it is sufficient to show that $\partial_t \mathcal{U}$ is bounded. Since $\mathcal{U}$ is a multiplication in time it is well known that

$$\|\mathcal{U}(t)\| \leq \sup_t \|\mathcal{U}(t)\| = \|\mathcal{U}\| \leq \sum_{k\in\mathbb{Z}} \|\mathcal{U}_{k,0}\|$$

where

$$\mathcal{U}_{m_1,n_1} := P^{(1)}_{m_1}\mathcal{U}P^{(1)}_{n_1} \quad \text{with} \quad P^{(1)}_{m_1} := (\cdot,\chi_{m_1})\chi_{m_1}.$$

Assume now that $H(0)$ is discrete with the following spectral decomposition

$$H(0) = \sum_{m_2\in I\!N} E_{m_2}P^{(2)}_{m_2}$$

where E_{m_2} and $P^{(2)}_{m_2}$ denote the eigenvalues and the associated eigenprojectors resp.. Notice that

$$M_{m_2} := \dim P_{m_2} < \infty$$

since $H(0)$ is discrete. We can further estimate the norm of bounded operators on $\mathcal{H}$ by the Schur-Holmgrem norm associated to the decomposition of $\mathcal{H}$ by the family $\{P^{(2)}_{n_2}\}$:

$$\|\mathcal{U}\| \le \sup_{m_2}\sum_{k\in Z\!\!Z}\sum_{n_2\in I\!N}\|\mathcal{U}_{k,m_2,n_2}\| \qquad \mathcal{U}_{k,m_2,n_2} := P^{(2)}_{m_2}\mathcal{U}_{k,0}P^{(2)}_{n_2}$$

which is just the type of norm used in [DLSV] where the proof of following theorem may be found. To state this theorem we define the norm on $\mathcal{B}(\mathcal{K})$

$$\|X\|_r := \sup_{m_2}\sum_{k\in Z\!\!Z}\sum_{n_2\in I\!N}\|P^{(2)}_{m_2}\mathcal{U}_{k,0}P^{(2)}_{n_2}\|\max\{|k|^r,1\}, \quad r\ge 0.$$

Theorem DLSV. *Let* $\omega_0>0$, $\Omega_0 := [\frac{8}{9}\omega_0, \frac{9}{8}\omega_0]$*; assume*

$$\exists\sigma>0, \qquad \frac{1}{(\Delta E_\sigma)^\sigma} := \sum_{m_2\ne n_2}\frac{M_{m_2}M_{n_2}}{|E_{m_2}-E_{n_2}|^\sigma} < \infty. \tag{16}$$

and let $\Delta_0 := \min_{m_2\ne n_2}|E_{m_2}-E_{n_2}|$.
Then, $\forall r>\sigma+\frac{1}{2}$, $\quad \exists C_1>0$ and $C_2(\sigma,r)>0$, *such that*

$$\|V\|_r < \min\left\{\frac{4\Delta_0}{C_1}, \frac{\omega_0}{C_1}, \frac{\omega_0}{C_2}\left(\frac{\Delta E_\sigma}{\omega_0}\right)^\sigma\right\} \tag{17}$$

implies

$$\exists\Omega_\infty\subset\Omega_0, \quad \text{with} \quad \frac{|\Omega_\infty|}{|\Omega_0|} \ge 1 - \frac{\|V\|_r}{\frac{\omega_0}{C_2}\left(\frac{\Delta E_\sigma}{\omega_0}\right)^\sigma}$$

so that K *is pure point for all* $\omega\in\Omega_\infty$. $|\Omega_\star|$ *denotes the Lebesgue measure of* $\Omega_\star$. *More precisely there exists a unitary operator* $\mathcal{U}$ *so that* $K=\mathcal{U}(K_0+G)\mathcal{U}^\star$

$$\sup_{m\in I\!N}\sum_{k\in Z\!\!Z}\sum_{n\in I\!N}|\mathcal{U}_{k,m,n}|\max\{|k|^{r-\sigma-\frac{1}{2}},1\} < \infty$$

where G is a bounded selfadjoint operator which commutes with D and $H(0)$.

We may state now the

Theorem 4. *Assume that $H(0)$ is discrete and simple and fulfills the growing gap condition (16). Assume also that $V(t) := H(t) - H(0)$ is symmetric and strongly C^r as a function of t with $r > \sigma + \frac{3}{2}$ and verifies (17). Assume finally that the frequency ω of V is in the set Ω_∞ of the above theorem. Then the conclusions of theorem 2 are valid.*

5 Two examples

We start by the quantum Fermi accelerator i.e. a free quantum particle in one dimensional pulsating box $(0, a(t))$ with period $T_a > 0$. The Floquet Hamiltonian of such a system may be cast into the form (see e.g. [Se, DS2] for details)

$$K := -i\partial_t + H(t), \quad H(t) := -\partial_x^2 + V(h(t), x), \quad V(t,x) := \frac{1}{4}\ddot{a}a^3x^2$$

acting in $\mathcal{K} := L^2(0,T) \otimes L^2(0,1)$ where h denotes the inverse function of $f : t \mapsto \int_0^t a^{-2}(s)ds$ and $T := f(T_a)$. Assume now that a is in $C^r(\mathbb{R}, \mathbb{R}_+ \setminus \{0\})$ with $r > 9/2$, it follows that V is in $C^{r-2}(\mathbb{R}, \mathcal{B}(L^2(0,1))$ in norm and the norm $\|V\|_{r-2}$ is finite. Also if $\ddot{a}a^3$ is small enough, $H(0)$ is discrete, simple and (16) is true with any $\sigma > 1$. Thus we may apply theorem 4 since $r - 2 > \sigma + 3/2$ for an appropriate σ.

The second example concerns the pulsed N-dimensional quantum top. Let $H(t) = -\Delta_{LB} + V(t,x)$ where Δ_{LB} denotes the Laplace-Beltrami operator on S^N, the N-dimensional unit sphere, and $(t,x) \mapsto V(t,x)$ is C^4 in $x \in S^N$ and C^r in t with $r > 2N + \frac{3}{2}$, see [DLSV] for more details. Again theorem 4 applies here for V small enough and ad hoc frequencies.

6 Concluding remarks

Given an arbitrary $r > 0$ we can show that the $\sup_{t\in\mathbb{R}_+} |(H(t)^r\psi(t), \psi(t))|$ is finite by requiring enough regularity of the eigenvectors of the Floquet Hamiltonian with respect to the time variable. Also one could get estimates on transition probabilities of the type $\|P(\Delta)U(t,0)P(\Delta')\| \leq C_r \mathrm{dist}\,(\Delta, \Delta')^{-r}$ where C_r is a constant which does not depend on Δ and Δ'. It would be useful for applications to treat the case of unbounded perturbations $V(t)$. We shall consider these extensions in a further publication.

As can be seen by a close look at the proof of theorem DLSV which is the key for theorem 4, the allowed frequencies are *non resonant* in the sense that all eigenvalues of $K_0 = D + H(0)$, namely $K_{0,m} := \omega m_1 + E_{m_2}$, $m := (m_1, m_2) \in \mathbb{Z} \times \mathbb{N}$, are such that $K_{0,m} = K_{0,n}$ iff $m = n$. What happen concerning this dynamical localization property for a pure point Floquet Hamiltonian, $K = K_0 + V$, with a resonant K_0, is still an open question. In particular whether one can find a pure point K with quantum trajectory which are not dynamically localized like in the example of [RiJLSi] is a challenge.

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