EDGE CURRENTS AND EIGENVALUE ESTIMATES FOR MAGNETIC BARRIER SCHRÖDINGER OPERATORS

NICOLAS DOMBROWSKI, PETER D. HISLOP, AND ERIC SOCCORSI

Abstract. We study two-dimensional magnetic Schrödinger operators with a magnetic field that is equal to $b > 0$ for $x > 0$ and $-b$ for $x < 0$. This magnetic Schrödinger operator exhibits a magnetic barrier at $x = 0$. The unperturbed system is invariant with respect to translations in the y-direction. As a result, the Schrödinger operator admits a direct integral decomposition. We analyze the band functions of the fiber operators as functions of the wave number and establish their asymptotic behavior. Because the fiber operators are reflection symmetric, the band functions may be classified as odd or even. The odd band functions have a unique absolute minimum. We calculate the effective mass at the minimum and prove that it is positive. The even band functions are monotone decreasing. We prove that the eigenvalues of an Airy operator, respectively, harmonic oscillator operator, describe the asymptotic behavior of the band functions for large negative, respectively positive, wave numbers. We prove a Mourre estimate for perturbations of the magnetic Schrödinger operator and establish the existence of absolutely continuous spectrum in certain energy intervals. We prove lower bounds on magnetic edge currents for states with energies in the same intervals. We also prove that these lower bounds imply stable lower bounds for the asymptotic currents. Because of the unique, non-degenerate minimum of the first band function, we prove that a perturbation by a slowly decaying negative potential creates an infinite number of eigenvalues accumulating at the bottom of the essential spectrum from below. We establish the asymptotic behavior of the eigenvalue counting function for these infinitely-many eigenvalues below the bottom of the essential spectrum.

Contents

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1. Statement of the problem and results

We continue our analysis of the spectral and transport properties of perturbed magnetic Schrödinger operators describing electrons in the plane $(x, y) \in \mathbb{R}^2$ moving under the influence of a transverse magnetic field exhibiting a discontinuity along the line $x = 0$. The basic model consists of a transverse magnetic field that is constant in each half plane so that it is equal to b_+ for $x > 0$ and b_- for $x < 0$, with $b_- \neq b_+$. In [14], two of us studied the generalized Iwatsuka model for which $0 < b_- < b_+ < \infty$. In this paper, we study the case for which $b_+ = b > 0$, for $x > 0$, and $b_- = -b < 0$, for $x < 0$. We choose a gauge so that the corresponding vector potential has the form $(0, A_2(x, y))$. The second component of the vector potential $A_2(x, y)$ is obtained by integrating the magnetic field so that $A_2(x, y) = b|x|$, independent of y. The fundamental magnetic Schrödinger operator is:

$$H_0 := p_x^2 + (p_y - b|x|)^2, \ p_x := -i\partial/\partial x, \ p_y := -i\partial/\partial y, \tag{1.1}$$

defined on the dense domain $C_0^\infty(\mathbb{R}^2) \subset \mathrm{L}^2(\mathbb{R}^2)$. This operator extends to a nonnegative self-adjoint operator in $\mathrm{L}^2(\mathbb{R}^2)$.

The magnetic field is piecewise constant and equals $\pm b$ on the half-planes $\mathbb{R}_\pm^* \times \mathbb{R}$, where $\mathbb{R}_\pm^* := \mathbb{R}_\pm \backslash \{0\}$. The discontinuity in the magnetic field at $x = 0$ is called a *magnetic edge*. Classically, a particle moving within a distance of

$\mathcal{O}(b^{-1/2})$ of the edge moves in a *snake orbit* [20]. Half of a snake orbit lies in the half-plane $x > 0$, and the other half of the orbit lies in $x < 0$. We prove that the quantum model has current flowing along the magnetic edge at $x = 0$ and that the current is localized in a small neighborhood of size $\mathcal{O}(b^{-1/2})$ of $x = 0$.

1.1. **Notation.** We write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and norm on $\mathrm{L}^2(\mathbb{R}^2)$. The functions are written with coordinates (x, y), or, after a partial Fourier transform with respect to y, we work with functions $f(x, k) \in \mathrm{L}^2(\mathbb{R}^2)$. We often view these functions $f(x, k)$ on $\mathrm{L}^2(\mathbb{R}_x)$ as parameterized by $k \in \mathbb{R}$. In this case, we also write $\langle f(\cdot, k), g(\cdot, k) \rangle$ and $\|f(\cdot, k)\|$ for the inner product and related norm on $\mathrm{L}^2(\mathbb{R}_x)$. So whenever an explicit dependance on the parameter k appears, the functions should be considered on $\mathrm{L}^2(\mathbb{R}_x)$. We indicate explicitly in the notation, such as $\| \cdot \|_X$, for $X = \mathrm{L}^2(\mathbb{R})$, when we work on those spaces. We write $\| \cdot \|_\infty$ for $\| \cdot \|_{\mathrm{L}^\infty(X)}$ for $X = \mathbb{R}, \mathbb{R}_\pm,$ or $\mathbb{R}^2$. For a subset $X \subset \mathbb{R}$, we denote by X^* the set $X^* := X \backslash \{0\}$. Finally for all $n \in \mathbb{N}$ we put $\mathbb{N}_n := \{j \in \mathbb{N},\ j \leqslant n\} = \{0, 1, \dots, n\}$.

1.2. **Fiber operators and reflection symmetry.** Due to the translational invariance in the y-direction, the operator H_0 on $\mathrm{L}^2(\mathbb{R}^2)$ is unitarily equivalent to the direct integral of operators $h(k)$, $k \in \mathbb{R}$, acting on $\mathrm{L}^2(\mathbb{R})$. This reduction is obtained using the partial Fourier transform with respect to the y-coordinate and defined as

$$(\mathcal{F}u)(x, k) = \hat{u}(x, k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-iyk} u(x, y) dy, \quad (x, k) \in \mathbb{R}^2.$$

Then we have $\mathcal{F} H_0 \mathcal{F}^* = \widehat{H}_0$ where

$$\widehat{H}_0 := \int_{\mathbb{R}}^{\oplus} h(k) dk,$$

and the fiber operator $h(k)$ acting in $\mathcal{H} := \mathrm{L}^2(\mathbb{R})$ is

$$h(k) := p_x^2 + (k - b|x|)^2, \quad k \in \mathbb{R}.$$

Since the effective potential $(k - b|x|)^2$ is unbounded as $|x| \to \infty$, the self-adjoint fiber operators $h(k)$ have compact resolvent. Consequently, the spectrum of $h(k)$ is discrete. We write $\omega_j(k)$ for the eigenvalues listed in increasing order. They are all simple (see [13, Appendix: Proposition A.2]) and depend analytically on k. As functions of $k \in \mathbb{R}$, these functions are called the *band functions* or *dispersion curves* and their properties play an important role. For fixed $k \in \mathbb{R}$, we denote by $\psi_j(\cdot, k)$ the L^2-normalized eigenfunctions of $h(k)$ with eigenvalue $\omega_j(k)$. For any $k \in \mathbb{R}$, these eigenvalues $\omega_j(k)$ satisfy the eigenvalue equation:

$$h(k)\psi_j(x, k) = \omega_j(k)\psi_j(x, k), \ \psi_j(\cdot, k) \in \mathrm{L}^2(\mathbb{R}_x), \ \|\psi_j(\cdot, k)\| = 1. \qquad (1.2)$$

We choose all $\psi_j(\cdot, k)$ to be real, and $\psi_1(x, k) > 0$ for $x \in \mathbb{R}$ and $k \in \mathbb{R}$. The rank-one orthogonal projections $P_j(k) := \langle \cdot, \psi_j(\cdot, k) \rangle \psi_j(\cdot, k)$, $j \in \mathbb{N}^*$, depend analytically on k by standard arguments.

The full operator H_0 exhibits reflection symmetry with respect to $x = 0$. Let I_P be the parity operator:

$$(I_P f)(x, y) := f(-x, y), \tag{1.3}$$

so that $I_P^2 = 1$. The Hilbert space $\mathrm{L}^2(\mathbb{R}^2)$ has an orthogonal decomposition corresponding to the eigenspaces of I_P with eigenvalue ± 1. The Hamiltonian H_0 commutes with I_P so each eigenspace of I_P is an H_0-invariant subspace.

This symmetry passes to the fiber decomposition. We denote by $\mathcal{I}_P$ the restriction to $\mathrm{L}^2(\mathbb{R}_x)$ of the operator I_P defined in (1.3), so that

$$(\mathcal{I}_P f)(x, k) = f(-x, k).$$

For any $k \in \mathbb{R}$, we have $[h(k), \mathcal{I}_P] = 0$. Since the eigenvalues of $h(k)$ are simple, for each $k \in \mathbb{R}$, there is a map $\theta_j : \mathbb{R} \to \{\pm 1\}$ so that

$$(\mathcal{I}_P \psi_j)(x, k) = \theta_j(k) \psi_j(x, k), \quad k \in \mathbb{R}, \quad j \in \mathbb{N}^*,$$

as $\psi_j(\cdot, k)$ is $\mathrm{L}^2(\mathbb{R}_x)$-normalized and real-valued. We show that $\theta_j(k)$ is independent of k. Since the mapping $k \mapsto P_j(k)$, the orthogonal projector onto $\psi_j(\cdot, k)$, is analytic, it follows that $\theta_j(k) = \theta_j(0)$ for every $k \in \mathbb{R}$. Consequently, each eigenfunction $\psi_j(x, k)$ is either even or odd in x.

We have an $h(k)$-invariant decomposition $\mathrm{L}^2(\mathbb{R}_x) = \mathcal{H}_- \oplus \mathcal{H}_+$, according to the eigenvalues $\{-1, +1\}$ of the projection $(\mathcal{I}_P f)(x) = f(-x)$. From this then follows that $h(k) = h^+(k) \oplus h^-(k)$, where

$$h^{\pm}(k) := h(k)_{|\mathcal{H}_{\pm}}, \ \mathcal{H}_{\pm} := \{f \in \mathcal{H}, \ \mathcal{I}_P f = \pm f\}.$$

We analyze the spectrum of $h(k)$ by studying the spectrum of the restricted operators letting $\sigma(h^{\pm}(k)) := \{\omega_j^{\pm}(k), \ j \in \mathbb{N}^*\}$. The operators $h^{\pm}(k)$ are unitarily equivalent to the operator $-d^2/dx^2 + (bx - k)^2$ on $\mathrm{L}(\mathbb{R}_x^+)$ with Neumann, respectively, Dirichlet, boundary conditions at $x = 0$ for $h^+(k)$, respectively, for $h^-(k)$. As a consequence we have $\omega_j^+(k) < \omega_j^-(k)$ from the min-max principle. It follows from this, the fact that $\sigma(h(k)) = \sigma(h^+(k)) \cup \sigma(h^-(k))$ for every $k \in \mathbb{R}$, that we have

$$\omega_j^+(k) = \omega_{2j-1}(k), \ \omega_j^-(k) = \omega_{2j}(k), \ j \in \mathbb{N}^*.$$

1.3. **Effective potential.** The fiber operator $h(k)$ has an effective potential:

$$V_{eff}(x, k) := (k - b|x|)^2, \ x, k \in \mathbb{R}.$$

The properties of this potential determine those of the band functions. The following characteristics of the potential follow from the formula and are the basis of the analytical studies in section 2.

Positive $k > 0$. There are two minima of V_{eff} at $x_{\pm} := \pm k/b$. The potential consists of two parabolic potential wells centered at $x_{\pm}$ and has value $V_{eff}(0, k) = k^2$. As $k \to +\infty$, the potential wells separate and the barrier between the two minima grows to infinity.

Negative $k < 0$. The effective potential is a parabola centered at $x = 0$ and $V_{eff}(0, k) = k^2$ is the minimum. Consequently, as $k \to -\infty$, the minimum of this potential well goes to plus infinity.

1.4. **Band functions.** The behavior of the effective potential described above determines the behavior of the band functions. For $k > 0$, the symmetric double wells of V_{eff} indicate that there are two eigenvalues near each level of a harmonic oscillator Hamiltonian. The splitting of these eigenvalues is exponentially small in the tunneling distance in the Agmon metric between $x_\pm$ (see, for example, [12, chapter 3]). As $k \to +\infty$, we establish in Proposition 2.3 that this tunneling effect is suppressed and these two eigenvalues approach the harmonic oscillator eigenvalue exponentially fast. For $k < 0$, there is a single potential well with a minimum that goes to infinity as $k \to -\infty$. Hence, the band functions diverge to plus infinity in this limit. This is the content of Proposition 2.2. Several band functions along with the parabola $E = k^2$ are shown in Figure 1.

1.5. **Relation to edge conductance.** The Iwatsuka model (1.1) attracted the attention of physicists [20] because of the intriguing classical snake orbits. In a wider context, these models (see also [14]) indicate how the magnetic field can confine the electrons along the magnetic discontinuity to create a net edge current. In the present article, we prove the existence of quantum states for which the expectation of the current operator is bounded from below and we give an explicit lower bound. This establishes electronic transport along the edge. A complementary point of view was established by Dombrowski, Germinet, and Raikov [10]. They studied the quantization of the Hall edge conductance for a generalized family of Iwatsuka models including the model discussed here.

Let us recall that the Hall edge conductance is defined as follows. We consider the situation where the edge lies along the y-axis as discussed above. Let $I := [a, b] \subset \mathbb{R}$ be a compact energy interval. We choose a smooth decreasing function g so that supp $g' \subset [a, b]$. Let $\chi = \chi(y)$ be an x-translation invariant smooth function with supp $\chi' \subset [-1/2, 1/2]$. The edge Hall conductance is defined by

$$\sigma_e^I(H) := -2\pi \mathrm{tr}\ (g'(H) i[H, \chi]),$$

whenever it exists. The edge conductance measures the current across the axis $y = 0$ carried by quantum states with energies in the energy interval I. Let $\mathbb{P}_0(I)$ be the spectral projector for H_0 and the interval Δ. Roughly speaking, if the trace in the expression for $\sigma_e^I(H)$ is expanded in a basis for the spectral subspace $\mathbb{P}_0(I)\mathrm{L}^2(\mathbb{R}^2)$, our result Theorem 4.1 proves that the edge conductance is positive. Due to the similarity with the quantum Hall effect, more is known about the edge conductance.

Theorem 2.2 of [10] presents the quantization of edge currents for the generalized Iwastuka model. So not only is the edge conductance positive, it assumes only integer values times a universal constant. For this model, the magnetic field $b(x)$ is simply assumed to be monotone and to have values $b_\pm$ at $\pm\infty$. The energy interval I is assumed to satisfy the following condition. There are two nonnegative integers $n_\pm \geqslant 0$ for which

$$I \subset ((2n_- - 1)|b_-|, (2n_- + 1)|b_-|) \cap ((2n_+ - 1)|b_+|, (2n_+ + 1)|b_+|),\ n_\pm \neq 0. \quad (1.4)$$

If $n_\pm = 0$, the corresponding interval should be taken to be $(-\infty, |b_\pm|)$. Under condition (1.4), Dombrowski, Germinet, and Raikov [10] proved

$$\sigma_e^I(H) = (\text{sign } b_-)n_- - (\text{sign } b_+)n_+.$$

Applied to the model studied here where $b_+ > 0$ and $b_- = -b_+ < 0$, and under condition (1.4), we have

$$\sigma_e^I(H) = -(n_- + n_+).$$

In particular, if $b_+ = b > 0$, and $I \subset ((2n-1)b, (2n+1)b)$, we have $\sigma_e^I(H) = -2n$.

Hence, our results complement this result on the quantization of edge conductance by proving in sections 3 and 4 the existence and localization of edge currents for H_0 and its perturbations. Following the notation of those sections, we prove, roughly speaking, that there is a nonempty interval Δ between the Landau levels $(2n-1)b$ and $(2n+1)b$ and a finite constant $c_n > 0$, so that for any state $\psi = \mathbb{P}_0(\Delta)\psi$ (recall that $\mathbb{P}_0(\Delta)$ is the spectral projector for H_0 and the interval Δ), we have

$$\langle \psi, v_y \psi \rangle \geqslant \frac{c_n}{2} b^{1/2} \|\psi\|^2 > 0, \ v_y := -(p_y - b|x|).$$

This lower bound indicates that such a state ψ carries a nontrivial edge current for H_0. We prove that this estimate is stable for a family of magnetic and electric perturbations of H_0.

1.6. **Contents.** We present the properties of the band functions $\omega_j(k)$ for the unperturbed fiber operator $h(k)$ in section 2. The emphasis is on the behavior of the band functions as $k \to \pm\infty$. The basic Mourre estimate for the unperturbed operator H_0 is derived in section 3 and its stability under perturbations is proven. As a consequence, this shows that there is absolutely continuous spectrum in certain energy intervals. Existence, localization, and stability of edge currents is established in section 4. We also prove a lower bound on the asymptotic velocity. In section 5, we study perturbations by negative potentials decaying at infinity. We demonstrate that because of the positive effective mass, such potentials create infinitely-many eigenvalues that accumulate at the bottom of the essential spectrum from below. We establish the asymptotic behavior of the eigenvalue counting function for these eigenvalues.

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Remark 1. After completion of this work, we learned of a similar analysis of the band structure by Nicolas Popoff [17] in his 2012 thesis at the Université

Rennes I. We thank Nicolas for many discussions and for letting us use his graph in Figure 1.

Remark 2. After completing this paper, we discovered the paper "Dirichlet and Neumann eigenvalues for half-plane magnetic Hamiltonians", by V. Bruneau, P. Miranda, and G. Raikov [6]. Their Corollary 2.4, part (i), is similar to our Theorem 5.1.

2. Properties of the band functions

In this section, we prove the basic properties of the band functions $k \in \mathbb{R} \mapsto \omega_j(k)$. We have the basic identity:

$$\omega_j(k) = \langle \psi_j(\cdot,k), h(k)\psi_j(\cdot,k)\rangle.$$

According to section 1.2, the eigenfunctions of $h(k)$ are either **even** and lie in $\mathcal{H}_+$, or **odd** and lie in $\mathcal{H}_-$, with respect to the reflection $x \mapsto -x$. We label the states so that the eigenfunctions $\psi_{2j-1} \in \mathcal{H}_+$ and $\psi_{2j} \in \mathcal{H}_-$, for $j = 1, 2, 3, \ldots$ The restrictions of $h(k)$ to $\mathcal{H}_\pm$ are denoted by $h^\pm(k)$, with eigenvalues $\omega_j^+(k) = \omega_{2j-1}(k)$ and $\omega_j^-(k) = \omega_{2j}(k)$, respectively.

Following the qualitative description in section 1.3, we have the following asymptotics for the band functions. We will prove in Proposition 2.3 that when $k \to +\infty$, the band function satisfies $\omega_j(k) \to (2j-1)b$, whereas, in Proposition 2.2, as $k \to -\infty$, we have $\omega_j(k) \to +\infty$. Similar analysis of these dispersion curves was performed in [8].

Proposition 2.1. *The band functions $\omega_j(k)$ are differentiable and the derivative satisfies*

$$\omega_j'(k) = \frac{-2}{b}\left[(\omega_j(k) - k^2)\psi_j(0,k)^2 + \psi_j'(0,k)^2\right]. \tag{2.1}$$

As a consequence, we have a classification of states:

(1) **Odd states:** $\psi_{2j}(0,k) = 0$. *The band functions satisfy:*

$$(\omega_j^-)'(k) = \omega_{2j}'(k) = \frac{-2}{b}\psi_{2j}'(0,k)^2 < 0. \tag{2.2}$$

(2) **Even states:** $\psi_{2j-1}'(0,k) = 0$. *The band functions satisfy:*

$$(\omega_j^+)'(k) = \omega_{2j-1}'(k) = \frac{-2}{b}(\omega_{2j-1}(k) - k^2)\psi_{2j-1}(0,k)^2. \tag{2.3}$$

Proof. The Feynman-Hellmann Theorem gives us

$$\begin{aligned}\omega_j'(k) &= \int_{\mathbb{R}} 2(k - b|x|)\psi_j(x,k)^2 \, dx \\ &= \frac{-1}{b}\int_0^\infty \psi_j(x,k)^2 \frac{d}{dx}(k-bx)^2 \, dx + \frac{1}{b}\int_{-\infty}^0 \psi_j(x,k)^2 \frac{d}{dx}(k+bx)^2 \, dx.\end{aligned}$$

Integrating by parts, and using the ordinary differential equation (1.2), we obtain (2.1). Note that $\lim_{x\to\pm\infty} x^2\psi_j(x,k)^2 = 0$ since $\psi_j(\cdot,k)$ is in the domain of $h(k)$ (see [15, Lemma 3.5]). □

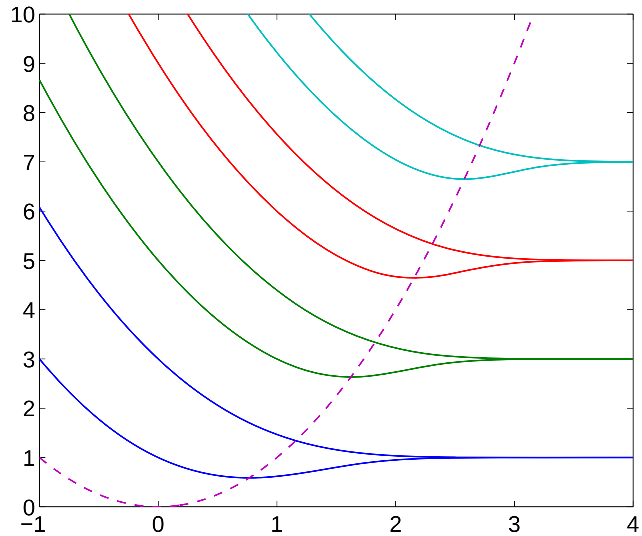

FIGURE 1. Approximate shape of the band functions $k \mapsto \omega_j(k)$, for $j = 1, \ldots, 8$, of the Iwatsuka Hamiltonian with $-b < 0 < b$ and $b = 1$. The dotted curve is $E = k^2$. Graph courtesy of N. Popoff.

Let us note that we cannot have both $\psi_j(0,k) = 0$ and $\psi_j'(0,k) = 0$. As consequences, the band functions for odd states are strictly monotone decreasing $\omega_{2j}'(k) < 0$. For even states, there is a minimum at $k = \kappa_j$ satisfying

$$\omega_{2j-1}(\kappa_j) = \kappa_j^2.$$

We will prove in Proposition 2.4 that this is the unique critical point of these band functions and that it is a non-degenerate minimum. This shows that there is an effective mass at this point. This is essential for the discussion in section 5.

2.1. **Band function asymptotics $k \to -\infty$.** As $k \to -\infty$, we will prove that the fiber Hamiltonian $h(k)$ is well approximated by an Airy operator

$$h_{\mathrm{Ai}}(k) := p_x^2 + 2b|k||x| + k^2, \tag{2.4}$$

in the sense that the band functions of $h(k)$ are close to the band functions of the Airy operator $h_{\mathrm{Ai}}(k)$. In order to establish this, let $Ai(x)$ be the standard Airy function whose zeros are located on the negative real axis. The Airy function satisfies the Airy ordinary differential equation:

$$Ai''(x) = xAi(x).$$

It follows that the scaled and translated Airy function $a_{\gamma,\sigma}(x) := Ai(\gamma x + \sigma)$ satisfies

$$(p_x^2 + \gamma^3 x)a_{\gamma,\sigma}(x) = -\gamma^2 \sigma a_{\gamma,\sigma}(x), \;\; \gamma, \sigma \in \mathbb{R}. \tag{2.5}$$

The model Airy Hamiltonian $h_{\mathrm{Ai}}(k)$ in (2.4) has discrete spectrum $\tilde{\omega}_j(k)$ and eigenfunctions $\tilde{\Psi}_j^{\mathrm{Ai}}(x,k)$ satisfying

$$h_{\mathrm{Ai}}(k)\tilde{\Psi}_j^{\mathrm{Ai}}(x,k) = \tilde{\omega}_j(k)\tilde{\Psi}_j^{\mathrm{Ai}}(x,k). \tag{2.6}$$

It follows from (2.5) that the eigenfunction $\tilde{\Psi}_j^{\mathrm{Ai}}(x,k)$ in (2.6) is a multiple of the scaled and translated Airy function $a_{\gamma,\sigma}$. The non-normalized solution $\tilde{\Psi}_j^{\mathrm{Ai}}(x,k)$ for the eigenvalue $\tilde{\omega}_j(k)$ is

$$\tilde{\Psi}_j^{\mathrm{Ai}}(x,k) = Ai\left((2b|k|)^{1/3}|x| + \frac{k^2 - \tilde{\omega}_j(k)}{(2b|k|)^{2/3}}\right),\ k<0,\ x \in \mathbb{R},$$

with an eigenvalue given by

$$\tilde{\omega}_j(k) = k^2 - (2b|k|)^{2/3}\sigma.$$

We determine σ as follows. The operator $h_{\mathrm{Ai}}(k)$ commutes with the parity operator $\mathcal{I}_P$, so its states are even or odd. The *odd eigenfunctions* $\Psi_j^{Ai,o}(x,k)$ of $h_{\mathrm{Ai}}(k)$ must satisfy $\Psi_j^{\mathrm{Ai,o}}(0,k) = 0$. Consequently, the $\mathrm{L}^2(\mathbb{R}_x)$-normalized odd eigenfunctions $\Psi_j^{\mathrm{Ai,o}}(x,k) = \tilde{\Psi}_{2j}^{Ai}(x,k)$ are given by

$$\Psi_j^{\mathrm{Ai,o}}(x,k) = C_{Ai,j}(b,k)(\mathrm{sign}\ x)Ai((2b|k|)^{1/3}|x| + z_{Ai,j}),\ \Psi_j^{\mathrm{Ai,o}}(0,k) = 0, \tag{2.7}$$

where $z_{Ai,j}$ is the j^{th} zero of $Ai(x)$ and the corresponding eigenvalue is

$$\tilde{\omega}_{2j}(k) = k^2 - (2b|k|)^{2/3} z_{Ai,j}.$$

The *even eigenfunctions* $\Psi_j^{\mathrm{Ai,e}}(x,k) = \tilde{\Psi}_{2j-1}^{Ai}(x,k)$ of $h_{\mathrm{Ai}}(k)$ must have a vanishing derivative at $x=0$ and are given by

$$\Psi_j^{\mathrm{Ai,e}}(x,k) = C_{Ai',j}(b,k)Ai((2b|k|)^{1/3}|x| + z_{Ai',j}),\ (\Psi_j^{\mathrm{Ai,e}})'(0,k) = 0, \tag{2.8}$$

and the corresponding eigenvalue is

$$\tilde{\omega}_{2j-1}(k) = k^2 - (2b|k|)^{2/3} z_{Ai',j},$$

where $z_{Ai',j}$ is the j^{th} zero of $Ai'(x)$. The normalization constant $C_{X,j}(b,k)$, for $X = Ai$ or Ai' is given by

$$C_{X,j}(b,k) := \left(\frac{(2b|k|)^{1/3}}{2c_{X,j}}\right)^{1/2}, \text{where } c_{X,j} := \int_0^\infty Ai(v + z_{X,j})^2\ dv. \tag{2.9}$$

We now obtain estimates on the band functions $\omega_j(k)$ as $k \to -\infty$.

Proposition 2.2. *For each $j \in \mathbb{N}^*$, as $k \to -\infty$, we have*

$$\|(h(k) - [k^2 - (2b|k|)^{2/3} z_{X,j}])\Psi_j^{\mathrm{Ai,u}}(\cdot,k)\| \leqslant \frac{b^{4/3}}{(2|k|)^{2/3}} D_{X,j}, \tag{2.10}$$

where the constant $D_{X,j}$, given in (2.11)*, is independent of the parameters (k,b), and $(X,u) = (Ai,e)$ or (Ai',o), for even or odd states, respectively. This immediately implies the eigenvalue estimate*

$$|\omega_j(k) - [k^2 - (2b|k|)^{2/3} z_{X,j}]| \leqslant \frac{b^{4/3}}{(2|k|)^{2/3}} D_{X,j},\ k \to -\infty.$$

Proof. In order to prove (2.10), we note that

$$h(k) - h_{\mathrm{Ai}}(k) = b^2 x^2,$$

so that with the definition of $\Psi_j^{\mathrm{Ai,u}}(x,k)$ in (2.7) for $u = o$ and (2.8) for $u = e$, and the normalization constant $C_{X,j}$ in (2.9), we have

$$\begin{aligned} \|[h(k) - h_{\mathrm{Ai}}(k)]\Psi_j^{\mathrm{Ai,u}}(\cdot,k)\|^2 &= \frac{2b^4}{(2|k|b)^{5/3}} C_{X,j}^2 \int_0^\infty v^4 Ai(v + z_{X,j})^2 \, dv \\ &= \frac{b^{8/3}}{(2|k|)^{4/3}} D_{X,j}^2, \end{aligned}$$

where the constant $D_{X,j}$, given by

$$D_{X,j} := \left(\frac{\int_0^\infty v^4 Ai(v + z_{X,j})^2 \, dv}{c_{X,j}} \right)^{1/2}, \tag{2.11}$$

is finite since $Ai(v) \sim e^{-v^{3/2}}$ as $v \to +\infty$. $\square$

2.2. **Band functions asymptotics $k \to +\infty$.** For $k \geqslant 0$, the effective potential consists of two double wells that separate as $k \to +\infty$. Consequently $\omega_j^+(k)$ approaches $\omega_j^-(k)$ as $k \to +\infty$. The eigenvalues of the double well potential consists of pairs of eigenvalues whose differences are super exponentially small as $k \to +\infty$. Thus, the effective Hamiltonian for $k = +\infty$ is the harmonic oscillator Hamiltonian:

$$h_{\mathrm{HO}}(k) := -\frac{d^2}{dx^2} + (bx - k)^2.$$

We let $\mathfrak{e}_0(b) := 0$ and $\mathfrak{e}_j(b) := (2j-1)b$, for every $j \in \mathbb{N}^*$, denote the energy levels of the harmonic oscillator. Let $\Psi_j^{\mathrm{HO}}(k)$ denote the j^{th} normalized eigenfunction of the harmonic oscillator so that $h_{\mathrm{HO}}(k)\Psi_j^{\mathrm{HO}}(k) = \mathfrak{e}_j(b)\Psi_j^{\mathrm{HO}}(k)$. It can be explicitly expressed as

$$\Psi_j^{\mathrm{HO}}(x,k) := \frac{1}{(2^j j!)^{1/2}} \left(\frac{b}{\pi}\right)^{1/4} \mathrm{e}^{-b/2(x-k/b)^2} H_j(b^{1/2}(x - k/b)), \tag{2.12}$$

where H_j is the j^{th} Hermite polynomial.

Proposition 2.3. *For each $j \in \mathbb{N}$, there exists a constant $0 < C_j < \infty$, depending only on j, so that for $k \geqslant 0$, we have,*

$$\|(h(k) - \mathfrak{e}_j(b))\Psi_j^{\mathrm{HO}}(\pm x; k)\| \leqslant C_j b \mathrm{e}^{-k^2/(4b)}. \tag{2.13}$$

This immediately implies the eigenvalue estimate

$$0 < \mp(\omega_j^{\pm}(k) - \mathfrak{e}_j(b)) \leqslant C_j b \mathrm{e}^{-\frac{k^2}{4b}}, \ k \geqslant \kappa_j, \tag{2.14}$$

and the difference of the two eigenvalues is bounded as

$$0 \leqslant \omega_j^-(k) - \omega_j^+(k) \leqslant 2C_j b \mathrm{e}^{-\frac{k^2}{4b}}, \ k \geqslant \kappa_j. \tag{2.15}$$

Proof. 1. Since $\Psi_j^{\rm HO}(k)$ is the eigenfunction of the harmonic oscillator Hamiltonian, we have for all $x \in \mathbb{R}$,

$$(h(k) - \mathfrak{e}_j(b))\Psi_j^{\rm HO}(\pm x, k) = ((bx \pm k)^2 - (bx \mp k)^2)\chi_{\mathbb{R}_\mp}(x)\Psi_j^{\rm HO}(\pm x, k),$$

so that for any $k \geqslant 0$, we have

$$\|(h(k) - \mathfrak{e}_j(b))\Psi_j^{\rm HO}(\pm x, k)\| \leqslant \|(bx \mp k)^2 \Psi_j^{\rm HO}(\pm x, k)\|_{{\rm L}^2(\mathbb{R}_\mp)}. \tag{2.16}$$

Here χ_I stands for the characteristic function of $I \subset \mathbb{R}$. From (2.16), the identity

$$\|(bx \mp k)^2 \Psi_j^{\rm HO}(\pm x, k)\|_{{\rm L}^2(\mathbb{R}_\mp)} = \|(bx - k)^2 \Psi_j^{\rm HO}(x, k)\|_{{\rm L}^2(\mathbb{R}_-)},$$

and (2.12), it follows that

$$\|(h(k) - \mathfrak{e}_j(b))\Psi_j^{\rm HO}(\pm x, k)\| \leqslant c_j b {\rm e}^{-k^2/(4b)},\ k \geqslant 0, \tag{2.17}$$

for some constant $c_j > 0$ depending only on j.

2. Let $(\Psi_j^{\rm HO})^\pm(x,k) := (\Psi_j^{\rm HO}(x,k) \pm \Psi_j^{\rm HO}(-x,k))/2 \in \mathcal{H}_\pm$. In light of (2.17) we have

$$\|(h(k) - \mathfrak{e}_j(b))(\Psi_j^{\rm HO})^\pm(k)\|_{\mathcal{H}} \leqslant c_j b {\rm e}^{-k^2/(4b)},\ k \geqslant 0. \tag{2.18}$$

Further since

$$\|(\Psi_j^{\rm HO})^\pm(k)\|^2 = \left(1 \pm \int_{\mathbb{R}} \Psi_j^{\rm HO}(x,k)\Psi_j^{\rm HO}(-x,k)dx\right)/2,$$

with

$$\left|\int_{\mathbb{R}} \Psi_j^{\rm HO}(x,k)\Psi_j^{\rm HO}(-x,k)dx\right| \leqslant \tilde{c}_j {\rm e}^{-k^2/(4b)},$$

for some constant $\tilde{c}_j > 0$ depending only on j, we deduce from (2.18) that

$$\mathrm{dist}(\sigma(h^\pm(k)), \mathfrak{e}_j(b)) \leqslant C_j b {\rm e}^{-k^2/(4b)},\ k \geqslant 0, \tag{2.19}$$

where $C_j > 0$ depends only on j.

3. As $\omega_j^-(k) > \mathfrak{e}_j(b)$ for each $k \in \mathbb{R}$, from the minimax principle, the result (2.14) for $\omega_j^-(k)$ follows readily from (2.19). The case of ω_j^+ is more complicated. In the section 2.3, we prove in the derivation of Proposition 2.4 that the band function $\omega_j^+(k)$ has a unique absolute minimum at a value $\kappa_j \in (0, \mathfrak{e}_{2j-1}(b)^{1/2})$. Furthermore, $\omega_j^+(\kappa_j) \in (\mathfrak{e}_{j-1}(b), \mathfrak{e}_j(b))$. We also prove that $(\omega_j^+)'(k) < 0$ for $k < \kappa_j$ and $(\omega_j^+)'(k) > 0$ for $k > \kappa_j$. The facts that the analytic band function is monotone increasing for $k > \kappa_j$ and converges to $\mathfrak{e}_j(b)$ as $k \to \infty$ due to (2.19) imply the result (2.14) for $\omega_j^+(k)$. ☐

2.3. **Even band functions $\omega_j^+(k)$: the effective mass.** We prove that the even states in $\mathcal{H}_+$, with band functions $\omega_j^+(k) = \omega_{2j-1}(k)$, have a unique positive minimum at κ_j. We prove that the even band function $\omega_j^+(k)$ is concave at κ_j. This convexity means that there is a positive effective mass. This positive effective mass plays an important role in the perturbation theory and creation of the discrete spectrum discussed in section 5.

Proposition 2.4. *The band functions $\omega_j^+(k) = \omega_{2j-1}(k)$, corresponding to the even states of $h(k)$, each have a unique extremum $\mathcal{E}_j \in (\mathfrak{e}_{j-1}(b), \mathfrak{e}_j(b))$ that is a strict minimum. The minimum is attained at a single point $\kappa_j \in (0, \mathfrak{e}_{2j-1}(b)^{1/2})$. This point is the unique real solution of $\omega_{2j-1}(k) - k^2 = 0$, and $\mathcal{E}_j = \kappa_j^2$. The concavity of the band function at κ_j is strictly positive and given by:*

$$(\omega_j^+)''(\kappa_j) = \omega_{2j-1}''(\kappa_j) = \frac{4\kappa_j}{b}\psi_{2j-1}(0,\kappa_j)^2 > 0. \tag{2.20}$$

We also have $\pm(\omega_j^+)'(k) < 0$ for $\pm(k - \kappa_j) < 0$.

Proof. 1. We first prove that there exists a unique minimum for the band function. The Feynman-Hellmann formula yields

$$(\omega_j^+)'(k) = -2\int_{\mathbb{R}} (b|x| - k)\psi_j^+(x,k)^2 dx,\ k \in \mathbb{R}. \tag{2.21}$$

Next, recalling (2.3), we get that

$$(\omega_j^+)'(k) = \frac{2}{b} f_j^+(k)\psi_j^+(0,k)^2,\ f_j^+(k) := k^2 - \omega_j^+(k), \tag{2.22}$$

since $\psi_j^+(0,k) \neq 0$ and $(\psi_j^+)'(0,k) = 0$. Moreover, taking into account that $h(0) = h_{\mathrm{HO}}(0)$ we see that

$$\omega_j^+(k) \leqslant \omega_j^+(0) = \mathfrak{e}_{2j-1}(b),\ k \in \mathbb{R}^+, \tag{2.23}$$

as $h(k) \leqslant h_{\mathrm{HO}}(k)$ in this case. Therefore we have $f_j^+(0) = -\mathfrak{e}_{2j-1}(b) < 0$ and $f_j^+(k) > 0$ for all $k > \mathfrak{e}_{2j-1}(b)^{1/2}$ from (2.23). The function f_j^+ is continuous in $\mathbb{R}$ hence there exists $\kappa_j \in (0, \mathfrak{e}_{2j-1}(b)^{1/2})$ such that $f_j^+(\kappa_j) = 0$. Moreover, f_j^+ being real analytic, the set $\{t \in \mathbb{R},\ f_j^+(t) = 0\}$ is at most discrete so we may assume without loss of generality that κ_j is its smallest element.
2. We next prove that $\omega_j^+(k)$ is decreasing for $k < \kappa_j$ and increasing for $k > \kappa_j$. It follows from (2.21) that $(\omega_j^+)'(k) < 2k$. Integrating this inequality over the interval $[\kappa_j, k]$, we obtain

$$\omega_j^+(k) < \omega_j^+(\kappa_j) + \int_{\kappa_j}^k 2t dt = \omega_j^+(\kappa_j) + (k^2 - \kappa_j^2),\ k > \kappa_j,$$

and hence $f_j^+(k) > f_j^+(\kappa_j)$ for all $k > \kappa_j$. This result with the fact that $f_j^+(\kappa_j) = 0$ and (2.22) imply that $(\omega_j^+)'(k) > 0$ for $k > \kappa_j$.
3. To study the concavity of the band function and establish (2.20), we differentiate (2.22) with respect to k and obtain

$$(\omega_j^+)''(k) = \frac{2}{b}\left(2f_j^+(k)\psi_j^+(0,k)\partial_k\psi_j^+(0,k) - [(\omega_j^+)'(k) - 2k]\psi_j^+(0,k)^2\right),\ \ k \in \mathbb{R}. \tag{2.24}$$

We evaluate (2.24) at κ_j, recalling that $f_j^+(\kappa_j) = 0$ and that $(\omega_j^+)'(\kappa_j) = 0$, in order to obtain (2.20).
4. We turn now to proving that $\mathcal{E}_j(b) \in (\mathfrak{e}_{j-1}(b), \mathfrak{e}_j(b))$. Since $(\omega_j^+)'(k) > 0$ for all $k \geqslant \kappa_j$ from Step 2 it follows readily from (2.14) that $\omega_j^+(\kappa_j) < \mathfrak{e}_j(b)$.

Further it is clear that $\omega_1^+(\kappa_1) > 0$ and we have in addition

$$\omega_j^+(k) = \omega_{2j-1}(k) > \omega_{2(j-1)}(k) = \omega_{j-1}^-(k) > \mathfrak{e}_{j-1}(b),\ k \in \mathbb{R},\ j \geqslant 2,$$

so the result follows. $\square$

In contrast to the even bands, for which the band function only asymptotically attains its minimum, the minimum of any odd band is attained at a finite wave number κ_j. The minimum of the first band $\mathcal{E}_1 := \omega_1(\kappa_1)$ is also the bottom of the spectrum of H_0. The fact that the first band has a unique, non-degenerate minimum at κ_1 has important consequences for the perturbation theory of H_0 as we will show in section 5. In light of Proposition 2.4, we say that the Hamiltonian H_0 has an effective mass $\beta_1 := \omega_1''(\kappa_1)/2 > 0$ at $k = \kappa_1$, borrowing this term from the solid state physics.

2.4. **Odd band functions $\omega_j^-(k)$: strict monotonicity.**
The behavior of the odd band functions is much simpler.

Proposition 2.5. *The odd band functions $\omega_j^-(k) = \omega_{2j}(k)$ are strictly monotone decreasing functions of $k \in \mathbb{R}$:*

$$(\omega_j^-)'(k) < 0,\ k \in \mathbb{R}.$$

Proof. Let us first recall from (2.1) of Proposition 2.1 that for all $k \in \mathbb{R}$ we have the formula

$$(\omega_j^\pm)'(k) = -\frac{2}{b}\left((\omega_j^\pm(k) - k^2)\psi_j^\pm(0,k)^2 + (\psi_j^\pm)'(0,k)^2\right). \tag{2.25}$$

Bearing in mind that $\psi_k^-(0,k) = 0$ and $(\psi_k^-)'(0,k) \neq 0$, the result follows immediately from (2.25). $\square$

2.5. **Absolutely continuous spectrum for H_0.**
The spectrum of H_0 is the closure of the union of the ranges of the band functions $\sigma(H_0) = \overline{\cup_{j\geqslant 1}\omega_j(\mathbb{R})} = [\omega_1(\kappa_1), \infty)$. The band functions are analytic and nonconstant by Proposition 2.4 for even states, and Proposition 2.5 for odd states. Consequently, from [19, Theorem XIII.86], the spectrum of H_0 is purely absolutely continuous.

3. Mourre estimates, perturbations, and stability of the absolutely continuous spectrum

In this section we study the spectrum of the operator H_0 and its perturbations using a Mourre estimate. For the unperturbed operator H_0, we prove a Mourre estimate using the fiber operator $h(k)$. This implies a lower bound on the velocity operator for certain states proving the existence of edge currents. We prove that this estimate is stable with respect to a class of perturbations.

3.1. **Mourre estimate for H_0.**
For all $E \in \mathbb{R}$ and all $\delta > 0$, we denote by $\Delta_E(\delta) := [E - (\delta/2)b, E + (\delta/2)b]$, the interval of width δ centered at E.

Lemma 3.1. *Let $n \in \mathbb{N}^*$, $E \in (\mathfrak{e}_n(b), \mathcal{E}_{n+1})$ and let $d_n(E)$ be the distance between E/b and the set $\{\mathfrak{e}_n(1), \mathcal{E}_{n+1}(1)\}$), i.e.*

$$d_n(E) := \max\left((E/b) - \mathfrak{e}_n(1), \mathcal{E}_{n+1}(1) - (E/b)\right).$$

Then there exists a constant $\delta_0 = \delta_0(E) \in (0, d_n(E))$*, independent of* b*, satisfying*

$$\omega_j^{-1}(\Delta_E(2\delta_0)) = \emptyset,\ j \geqslant 2n+1, \tag{3.1}$$

and

$$\omega_i^{-1}(\Delta_E(2\delta_0)) \cap \omega_j^{-1}(\Delta_E(2\delta_0)) = \emptyset,\ 1 \leqslant i \neq j \leqslant 2n. \tag{3.2}$$

Moreover, for every $j \in \mathbb{N}_{2n}^*$ *there is a constant* $c_{n,j} = c_{n,j}(E) > 0$*, independent of* b*, such that we have*

$$-\omega_j'(k) \geqslant c_{n,j} b^{1/2},\ k \in \omega_j^{-1}(\Delta_E(2\delta_0)). \tag{3.3}$$

Proof. 1. First (3.1) follows readily from Proposition 2.4 and the fact that $\Delta_E(2\delta_0) \cap [\mathcal{E}_{n+1}(b), +\infty) = \emptyset$ for all $\delta_0 \in (0, d_n(E))$ since $E + \delta_0 b < \mathcal{E}_{n+1}(b)$.
2. Next we notice that $h(k)$ is unitarily equivalent to the operator $b\check{h}(k/b^{1/2})$, where

$$\check{h}(q) := -\frac{d^2}{dt^2} + (|t| - q)^2,\ q \in \mathbb{R},$$

is defined on the dense domain $\mathrm{C}_0^\infty(\mathbb{R}) \subset \mathrm{L}^2(\mathbb{R})$. More precisely it holds true that $\mathcal{V}_b h(k) \mathcal{V}_b^* = b\check{h}(k/b^{1/2})$, where

$$(\mathcal{V}_b \psi)(x) := b^{-1/4} \psi(x/b^{1/2}),\ \psi \in \mathrm{L}^2(\mathbb{R}),$$

is easily seen to be a unitary transform in $\mathrm{L}^2(\mathbb{R})$. As a consequence we have

$$\omega_j(k) = b\breve{\omega}_j(k/b^{1/2}),\ k \in \mathbb{R},\ j \in \mathbb{N}^*, \tag{3.4}$$

where $\{\breve{\omega}_j(k)\}_{j=1}^\infty$ is the set of eigenvalues (arranged in increasing order) of $\check{h}(k)$. Let $a_{n,j}$, $j \in \mathbb{N}_{2n-1}^*$, be the unique real number obeying $\breve{\omega}_j(a_{n,j}) = \mathfrak{e}_n(1)$, set $a_{n,2n} := +\infty$, and denote by $\breve{\omega}_j^{-1}$ the function inverse to $\breve{\omega}_j : (-\infty, a_{n,j}) \to (\mathfrak{e}_n(1), +\infty)$. As the interval $[(E/b) - \delta_0, (E/b) + \delta_0] \subset (\mathfrak{e}_n(1), \mathcal{E}_{n+1}(1))$, it is in the domain of each function $\breve{\omega}_j^{-1}$, $j \in \mathbb{N}_{2n}^*$, and we have

$$\breve{\omega}_j^{-1}([(E/b) - \delta_0, (E/b) + \delta_0]) = [\breve{\omega}_j^{-1}((E/b) + \delta_0), \breve{\omega}_j^{-1}((E/b) - \delta_0)], \tag{3.5}$$

by Propositions 2.4 and 2.5. Further, since $\breve{\omega}_{j+1}^{-1}(E) > \breve{\omega}_j^{-1}(E)$ for all $j \in \mathbb{N}_{2n-1}^*$, the functions $\breve{\omega}_j^{-1}$ are continuous, and $2n - 1$ is finite, then there is necessarily $\delta_0 \in (0, d_n(E))$ such that we have

$$\breve{\omega}_{j+1}^{-1}((E/b) + \delta_0) > \breve{\omega}_j^{-1}((E/b) - \delta_0),\ j \in \mathbb{N}_{2n-1}^*.$$

This and (3.4)-(3.5) yields (3.2).
3. Finally, taking into account that $\check{h}(q)$ coincides with $h(q)$ in the particular case where $b = 1$, we deduce from Propositions 2.4 and 2.5 for any $\Delta \subset (\mathfrak{e}_n(1), \mathcal{E}_{n+1}(1))$ that

$$\inf_{q \in \breve{\omega}_j^{-1}(\Delta)} (-\breve{\omega}_j'(q)) = \check{c}_j(\Delta) > 0,\ j \in \mathbb{N}_{2n}^*, \tag{3.6}$$

where the constant $\check{c}_j(\Delta)$ is independent of b. Now (3.3) follows readily from (3.6) since $[(E/b) - \delta_0, (E/b) + \delta_0] \subset (\mathfrak{e}_n(1), \mathcal{E}_{n+1}(1))$ and

$$\inf_{k \in \omega_j^{-1}(\Delta_E(2\delta_0))} (-\omega_j'(k)) = b^{1/2} \inf_{q \in \breve{\omega}_j^{-1}([(E/b) - \delta_0, (E/b) + \delta_0])} (-\breve{\omega}_j'(q)),$$

according to (3.4). ∎

Let us now introduce the operator $M = M^* := -y$ defined originally on $C_0^\infty(\mathbb{R}^2)$. The operator M extends to a self-adjoint operator in $L^2(\mathbb{R}^2)$. Note that $C_0^\infty(\mathbb{R}^2)$ is dense in $\mathrm{Dom}(H_0)$ and hence that $\mathrm{Dom}(M) \cap \mathrm{Dom}(H_0)$ is dense in $\mathrm{Dom}(H_0)$.

Proposition 3.1. *Let $b > 0$, $n \in \mathbb{N}^*$, $E \in (\mathfrak{e}_n(b), \mathcal{E}_{n+1}(b))$ and assume that $\delta_0 \in (0, d_n(E))$ is chosen to satisfy* (3.1)-(3.2) *according to Lemma 3.1. Let $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \chi \subset \Delta_E(2\delta_0)$. Then there exists a constant $c_n = c_n(E) > 0$, independent of b, such that we have*

$$\chi(H_0)[H_0, iM]\chi(H_0) \geqslant c_n b^{1/2} \chi(H_0)^2, \tag{3.7}$$

as a quadratic form on $\mathrm{Dom}(M) \cap \mathrm{Dom}(H_0)$.

Proof. We get

$$[H_0, iM] = -2(p_y - b|x|), \tag{3.8}$$

on $\mathrm{Dom}(M) \cap \mathrm{Dom}(H_0)$. We recall the orthogonal projection $P_j(k)$ associated with $h(k)$ and defined by $P_j(k) := \langle \cdot, \psi_j(\cdot, k)\rangle \psi_j(\cdot, k)$, for all $j \in \mathbb{N}^*$. The commutator on the left in (3.8) fibers over $k \in \mathbb{R}$, so by a direct calculation, we find that

$$\chi(H_0)[H_0, iM]\chi(H_0) = -2\mathcal{F}^* \left(\sum_{j,m \in \mathbb{N}^*} \int_{\mathbb{R}}^{\oplus} \chi(\omega_j(k))\chi(\omega_m(k)) P_j(k)(k - b|x|) P_m(k) dk \right) \mathcal{F}.$$

Taking into account that $\operatorname{supp} \chi \subset \Delta_E(2\delta_0)$, we deduce from (3.1)-(3.2) that

$$\chi(H_0)[H_0, iM]\chi(H_0) = -2\mathcal{F}^* \left(\sum_{j=1}^{2n} \int_{\mathbb{R}}^{\oplus} \chi(\omega_j(k))^2 \langle \psi_j(\cdot, k), (k - b|x|)\psi_j(\cdot, k)\rangle P_j(k) dk \right) \mathcal{F},$$

whence

$$\chi(H_0)[H_0, iM]\chi(H_0) = \mathcal{F}^* \left(\sum_{j=1}^{2n} \int_{\mathbb{R}}^{\oplus} \chi(\omega_j(k))^2 (-\omega_j'(k)) P_j(k) dk \right) \mathcal{F}, \tag{3.9}$$

from the Feynman-Hellmann formula. In light of (3.3), we have

$$-\omega_j'(k)\chi(\omega_j(k))^2 \geqslant c_{n,j} b^{1/2} \chi(\omega_j(k))^2, \ j \in \mathbb{N}_{2n}^*,$$

so (3.9) yields

$$\chi(H_0)[H_0, iM]\chi(H_0) \geqslant c_n b^{1/2} \mathcal{F}^* \left(\sum_{j=1}^{2n} \int_{\mathbb{R}}^{\oplus} \chi(\omega_j(k))^2 P_j(k) dk \right) \mathcal{F} = c_n b^{1/2} \chi(H_0)^2,$$

where $c_n := \min_{j \in \mathbb{N}_{2n}^*} c_{n,j} > 0$. $\square$

As above, let $\mathbb{P}_0(I)$ denote the spectral projection of H_0 for the Borel set $I \subset \mathbb{R}$. Then by choosing χ in Proposition 3.1 to be equal to one on $\Delta_E(\delta_0)$ and multiplying (3.7) from both sides by $\mathbb{P}_0(\Delta_E(\delta_0))$, we obtain the following Mourre estimate for H_0:

$$\mathbb{P}_0(\Delta_E(\delta_0))[H_0, iM]\mathbb{P}_0(\Delta_E(\delta_0)) \geqslant c_n b^{1/2} \mathbb{P}_0(\Delta_E(\delta_0)). \tag{3.10}$$

3.2. **Edge currents for H_0.** We can prove the existence of edge currents for the unperturbed Hamiltonian H_0 based on the Mourre estimate (3.10). A state $\varphi \in \mathrm{L}^2(\mathbb{R}^2)$ carries an edge current of the Hamiltonian H if $J_y(\varphi) := \langle \varphi, v_y \varphi \rangle$ is strictly positive, where the velocity operator is $v_y = (i/2)[H, M]$.

Corollary 3.1. *Let b, n, E, and δ_0 be as in Proposition 3.1. Let $\varphi \in \mathrm{L}^2(\mathbb{R}^2)$ satisfy $\varphi = \mathbb{P}_0(\Delta_E(\delta_0))\varphi$. Then φ carries an edge current and the edge current is bounded below by*

$$J_y(\varphi) \geqslant \frac{c_n}{2} b^{1/2} \|\varphi\|^2, \tag{3.11}$$

where c_n is the constant defined in Proposition 3.1.

The proof of this corollary follows directly from (3.10) since for φ as in the corollary, we have $J_y(\varphi) = \langle \varphi, (1/2)\mathbb{P}_0(\Delta_E(\delta_0))[H_0, iM]\mathbb{P}_0(\Delta_E(\delta_0))\varphi \rangle$. The edge currents associated with H_0 and states φ as in Corollary 3.1 are also localized in a neighborhood of size roughly $b^{-1/2}$ about $x = 0$. This follows from Proposition 4.1.

3.3. **Stability of the Mourre estimate.** One of the main benefits of a local commutator estimate like (3.10) is its stability under perturbation. Namely we consider the perturbation of $H_0 = (-i\nabla - A_0)$, $A_0 = A_0(x, y) := (0, b|x|)$, by a magnetic potential $a(x, y) = (a_1(x, y), a_2(x, y)) \in \mathrm{W}^{1,\infty}(\mathbb{R}^2)$ and a bounded scalar potential $q(x, y) \in \mathrm{L}^\infty(\mathbb{R}^2)$. We prove that a Mourre inequality for the perturbed operator

$$H = H(a, q) := (-i\nabla - A_0 - a)^2 + q = (p_x - a_1)^2 + (p_y - b|x| - a_2)^2 + q, \tag{3.12}$$

remains true provided $\|a\|_{W^{1,\infty}(\mathbb{R}^2)}$ and $\|q\|_\infty$ are small enough relative to b. We preliminarily notice that

$$W = W(a) := H(a, 0) - H_0 = 2a \cdot (-i\nabla - A_0) - i(\nabla \cdot a) + a \cdot a, \tag{3.13}$$

with $\|(-i\nabla - A_0)\varphi\| = \langle H_0\varphi, \varphi \rangle^{1/2} \leqslant \lambda \|H_0 \varphi\| + \lambda^{-1}\|\varphi\|^2$ for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $\lambda > 0$, so we have

$$\|W\varphi\| \leqslant 2\lambda \|a\|_\infty \|H_0\varphi\| + (\lambda^{-1} + \|\nabla a\|_\infty + \|a\|_\infty^2)\|\varphi\|, \ \lambda > 0.$$

Taking $\lambda = 1/(4\|a\|_\infty)$ in the above inequality we find that W is H_0-bounded with relative bound smaller than one. In light of [19][Theorem X.12] the operator $H(a, 0) = H_0 + W$ is thus selfadjoint in $\mathrm{L}^2(\mathbb{R}^2)$ with same domain as H_0, and the same is true for $H = H(a, q) = H(a, 0) + q$ since $q \in \mathrm{L}^\infty(\mathbb{R}^2)$.

Proposition 3.2. *Let b, n, E and δ_0 be as in Proposition 3.1. Assume that $\delta = \delta(E) \in (0, \delta_0)$, $a \in \mathrm{W}^{1,\infty}(\mathbb{R}^2)$ and $q \in L^\infty(\mathbb{R}^2)$ verify*

$$F_{n,E}\left(\delta, \frac{\|q\|_\infty}{b}, \frac{\|a\|_\infty^2 + \|\nabla a\|_\infty}{b}\right) < \frac{1}{2}, \tag{3.14}$$

[1]*where*

$$F_{n,E}(\delta, \mathfrak{a}, \mathfrak{q}) := \left(\frac{f_n(\delta, \mathfrak{a}, \mathfrak{q})}{\delta_0}\right)^2 + \frac{2}{c_n}\left(\mathfrak{a}^{1/2} + (2n + 1 + f_n(\delta, \mathfrak{a}, \mathfrak{q}))^{1/2}\left(\frac{f_n(\delta, \mathfrak{a}, \mathfrak{q})}{\delta_0}\right)^{1/2}\right), \tag{3.15}$$

[1]Notice that the function $F_{n,E}$ depends on E through $\delta_0 = \delta_0(E)$.

f_n is given by (3.25) and c_n is the constant defined in Proposition 3.1. Then we have the following Mourre estimate

$$\mathbb{P}(\Delta_E(\delta))[H, iM]\mathbb{P}(\Delta_E(\delta)) \geqslant \frac{c_n}{2} b^{1/2}\mathbb{P}(\Delta_E(\delta)), \tag{3.16}$$

where $\mathbb{P}(I)$ denotes the spectral projection of H for the Borel set $I \subset \mathbb{R}$.

Proof. By combining the following decomposition of $\psi \in \mathbb{P}(\Delta_E(\delta))\mathrm{L}^2(\mathbb{R}^2)$ into the sum

$$\psi = \phi + \xi,\ \phi := \mathbb{P}_0(\Delta_E(\delta_0))\psi,\ \xi := \mathbb{P}_0(\mathbb{R} \setminus \Delta_E(\delta_0))\psi, \tag{3.17}$$

with the basic equality

$$[H, iM] = [H_0, iM] + 2a_2, \tag{3.18}$$

obtained through standard computations, we get that

$$\langle \psi, [H, iM]\psi\rangle = \langle \phi, [H_0, iM]\phi\rangle + 2\langle \psi, a_2\psi\rangle + C(\phi, \xi),$$

with

$$\begin{aligned} C(\phi,\xi) \quad := \quad & \int_{\mathbb{R}} \langle \hat{\xi}(\cdot,k), (k - b|x|)\hat{\xi}(\cdot,k)\rangle_{\mathrm{L}^2(\mathbb{R})} dk \\ & +2\mathrm{Re}\left(\int_{\mathbb{R}} \langle \hat{\phi}(\cdot,k), (k - b|x|)\hat{\xi}(\cdot,k)\rangle_{\mathrm{L}^2(\mathbb{R})} dk\right). \end{aligned}$$

This entails

$$\langle \psi, [H, iM]\psi\rangle \geqslant \langle \phi, [H_0, iM]\phi\rangle - 2\left(\|a\|_\infty\|\psi\| + \|(p_y - b|x|)\xi\|\right)\|\psi\|, \tag{3.19}$$

since

$$\|\psi\|^2 = \|\phi\|^2 + \|\xi\|^2,$$

as can be seen from the orthogonality of ϕ and ξ in $\mathrm{L}^2(\mathbb{R}^2)$, arising from (3.17). The first term in the right hand side of (3.19) is lower bounded by (3.10) as

$$\langle \phi, [H_0, iM]\phi\rangle \geqslant c_n b^{1/2}\|\phi\|^2, \tag{3.20}$$

and $\|(p_y - b|x|)\xi\|$ can be bounded above with the help of the estimate

$$\|(p_y - b|x|)\xi\|^2 \leqslant \langle \xi, H_0\xi\rangle = \langle \psi, H_0\xi\rangle = \langle H_0\psi, \xi\rangle = \langle (H - W - q)\psi, \xi\rangle,$$

giving

$$\|(p_y - b|x|)\xi\|^2 \leqslant \langle \xi, H_0\xi\rangle \leqslant ((E/b) + \delta + \mathfrak{q} + w)\, b\|\xi\|\|\psi\|, \tag{3.21}$$

where $\mathfrak{q} := \|q\|_\infty/b$ and $w := \|W\psi\|/(b\|\psi\|)$. Further we have

$$\|\xi\| \leqslant \frac{\delta + \mathfrak{q} + w}{\delta_0}\|\psi\|, \tag{3.22}$$

since $\|\xi\|^2 = \langle (H - E - W - q)\psi, (H_0 - E)^{-1}\xi\rangle$. In light of the right hand side of (3.21)-(3.22) we are thus left with the task of obtaining an upper bound on w. This can be done by combining the estimate

$$\|(-i\nabla - A_0)\psi\| = \langle H_0\psi, \psi\rangle^{1/2} \leqslant \|H_0\psi\|^{1/2}\|\psi\|^{1/2} \leqslant \|(H - W - q)\psi\|^{1/2}\|\psi\|^{1/2},$$

entailing $\|(-i\nabla - A_0)\psi\| \leqslant ((E/b) + \delta + \mathfrak{q} + w)^{1/2} b^{1/2}\|\psi\|$, with (3.13). We find out that $w \leqslant \mathfrak{a}^{1/2}\left(\mathfrak{a}^{1/2} + 2\left((E/b) + \delta + \mathfrak{q} + w\right)^{1/2}\right)$ with $\mathfrak{a} := (\|a\|_\infty^2 +$

$\|\nabla a\|_\infty)/b$, whence $w \leqslant 2\mathfrak{a}^{1/2}\left(3\mathfrak{a}^{1/2} + (E/b) + \delta + \mathfrak{q}\right)^{1/2})$. From this, the estimate $E \leqslant (2n+1)b$, arising from Proposition 2.4, and (3.21)-(3.22) then follows that

$$\|\xi\| \leqslant \frac{f_n(\delta, \mathfrak{a}, \mathfrak{q})}{\delta_0}\|\psi\|, \tag{3.23}$$

and

$$\|(p_y - b|x|)\xi\| \leqslant (2n+1+f_n(\delta,\mathfrak{a},\mathfrak{q}))^{1/2}\left(\frac{f_n(\delta,\mathfrak{a},\mathfrak{q})}{\delta_0}\right)^{1/2} b^{1/2}\|\psi\|, \tag{3.24}$$

where

$$f_n(\delta,\mathfrak{a},\mathfrak{q}) := \delta + \mathfrak{q} + 2\mathfrak{a}^{1/2}\left(3\mathfrak{a}^{1/2} + (2n+1+\delta+\mathfrak{q})^{1/2}\right). \tag{3.25}$$

Putting (3.19)-(3.20) and (3.23)-(3.24) together and recalling (3.15) we end up getting that

$$\langle \psi, [H, iM]\psi\rangle \geqslant c_n\left(1 - F_{n,E}(\delta,\mathfrak{q},\mathfrak{a})\right) b^{1/2}\|\psi\|^2,$$

so (3.16) follows readily from this and (3.14). □

3.4. **Absolutely continuous spectrum.** We now apply Proposition 3.2 to prove the existence of absolutely continuous spectrum for perturbed magnetic barrier operators. Using direct computation, we deduce from (3.8) and (3.18) that $[[H, iM], iM] = -2$. Hence the double commutator of H with M is bounded from $\mathrm{Dom}(H) = \mathrm{Dom}(H_0)$ to $\mathrm{L}^2(\mathbb{R}^2)$. Moreover, since $[H, iM]$ extends to a bounded operator from $\mathrm{Dom}(H_0)$ to $\mathrm{L}^2(\mathbb{R}^2)$, the Mourre estimate (3.16) combined with [7][Corollary 4.10] entails the following:

Corollary 3.2. *Let b, n, E and δ_0 be the same as in Proposition 3.1. Assume that $\delta \in (0, \delta_0)$, $q \in \mathrm{L}^\infty(\mathbb{R}^2)$ and $a \in \mathrm{W}^{1,\infty}(\mathbb{R}^2)$ satisfy (3.14). Then the spectrum of $H = H(a,q)$ in $\Delta_E(\delta)$ is absolutely continuous.*

Armed with Corollary 3.2 we turn now to proving the main result of this section.

Theorem 3.1. *Let $b > 0$, $n \in \mathbb{N}^*$, and let Δ be a compact subinterval of $(\mathfrak{e}_n(b), \mathcal{E}_{n+1}(b))$. Then there are two constants $\mathfrak{a}^* = \mathfrak{a}^*(n,\Delta) > 0$ and $\mathfrak{q}^* = \mathfrak{q}^*(n,\Delta) > 0$, both independent of b, such that for all $(a,q) \in \mathrm{W}^{1,\infty}(\mathbb{R}^2)\times\mathrm{L}^2(\mathbb{R}^2)$ verifying $\|a\|_\infty^2 + \|\nabla a\|_\infty \leqslant \mathfrak{a}^* b$ and $\|q\|_\infty \leqslant \mathfrak{q}^* b$, the spectrum of $H = H(a,q)$ in Δ is absolutely continuous.*

Proof. For every $E \in \Delta$ choose $\delta(E) \in (0, \delta_0(E))$, $\mathfrak{a}(E) > 0$ and $\mathfrak{q}(E) > 0$ such that

$$F_{n,E}(\delta(E), \mathfrak{a}(E), \mathfrak{q}(E)) < \frac{1}{2}, \tag{3.26}$$

where $F_{n,E}$ is defined in (3.15).

Since Δ is compact and $\Delta \subset \cup_{E\in\Delta}\Delta_E(\delta(E))$, there exists a finite set $\{E_j\}_{j=1}^N$ of energies in Δ such that

$$\Delta \subset \bigcup_{j=1}^N \Delta_{E_j}(\delta(E_j)). \tag{3.27}$$

Set $\mathfrak{a}^* := \min_{1\leqslant j\leqslant N} \mathfrak{a}(E_j) > 0$ and $\mathfrak{q}^* := \min_{1\leqslant j\leqslant N} \mathfrak{q}(E_j) > 0$. Since $F_{n,E_j}(\delta(E_j),\cdot,\cdot)$, $j=1,\dots,N$, is an increasing function of each of the two last variables taken separately, when the remaining one is fixed, we necessarily have $F_{n,E_j}(\delta(E_j),\mathfrak{a}^*,\mathfrak{q}^*) < 1/2$ by (3.26). Assume that $\|a\|_\infty^2+\|\nabla a\|_\infty \in [0,\mathfrak{a}^* b)$ and $\|q\|_\infty \in [0,\mathfrak{q}^* b)$. For every $j=1,\dots,N$, the spectrum of H in $\Delta_{E_j}(\delta(E_j))$ is thus absolutely continuous by Corollary 3.2 so the result follows from this and (3.27). $\square$

4. Edge currents: existence, stability, localization, and asymptotic velocity

A major consequence of the Mourre estimate in Proposition 3.1 for the unperturbed operator H_0 is the lower bound on the edge current carried by certain states given in Corollary 3.1. Because of the stability result for the Mourre estimate for the perturbed operator $H(a,q)$ in Proposition 3.2, we prove in this section that edge currents are stable under perturbations. We then prove that these currents are well-localized in a strip of width $\mathcal{O}(b^{-1/2})$ about $x=0$. Finally, we prove that the asymptotic velocity is bounded from below demonstrating that the edge currents persist for all time.

4.1. **Existence and stability of edge currents.** For the perturbed operator $H=H(a,q)$, the y-component of the velocity operator is

$$v_{y,a,q} := (1/2)[H,iM] = -(p_y - b|x|) + a_2,\ M=-y, \tag{4.1}$$

according to (3.8) and (3.18). A state $\varphi \in \mathrm{L}^2(\mathbb{R}^2)$ carries an edge current if

$$J_{y,a,q}(\varphi) := \langle \varphi, v_{y,a,q}\varphi\rangle \geqslant c\|\varphi\|^2, \tag{4.2}$$

for some constant $c>0$. For notational simplicity we write v_y (resp. J_y) instead of $v_{y,0,0}$ (resp. $J_{y,0,0}$) in the particular case of the unperturbed operator H_0 corresponding to $a=0$ and $q=0$. We consider states in the range of the spectral projector $\mathbb{P}_0(\cdot)$ for H_0, and in the range of the spectral projector $\mathbb{P}_{a,q}(\cdot)$ for $H=H(a,q)$, and energy intervals as in (3.10) for H_0, and in (3.16) for $H_{a,q}$. We then deduce from (4.1)-(4.2) the existence of edge currents for the operator H_0 and $H_{a,q}$, respectively. We recall Corollary 3.1 in the first part of the following theorem.

Theorem 4.1. *Let b, n, E, and δ_0 be as in Proposition 3.1.*

(1) *Let $\varphi \in \mathrm{L}^2(\mathbb{R}^2)$ satisfy $\varphi = \mathbb{P}_0(\Delta_E(\delta_0))\varphi$. Then φ carries an edge current obeying*
$$J_y(\varphi) \geqslant \frac{c_n}{2} b^{1/2}\|\varphi\|^2,$$
where c_n is the constant defined in Proposition 3.1.

(2) *Let $\delta \in (0,\delta_0)$ and assume that $(a,q) \in W^{1,\infty}(\mathbb{R}^2)\times \mathrm{L}^\infty(\mathbb{R}^2)$ verifies the condition (3.14), where $(c_n/2)$ is substituted for c_n in the definition (3.15). Then every state $\varphi \in \mathbb{P}_{a,q}(\Delta_E(\delta))\mathrm{L}^2(\mathbb{R}^2)$ carries an edge current and we have the lower bound*
$$J_{y,a,q}(\varphi) \geqslant \frac{c_n}{4} b^{1/2}\|\varphi\|^2. \tag{4.3}$$

4.2. **Localization of edge currents.** We establish the localization of the edge currents described in Theorem 4.1 using a method introduced by Iwatsuka [15, section 3]. We refer the reader to section 3.1 for the definitions of the various quantities appearing in the following proposition.

Proposition 4.1. *Let n, E, and Δ_0 be as in Proposition 3.1 and choose $\delta = \delta(E) \in (0, \delta_0)$ in accordance with condition (3.14). Let $\varphi \in \mathrm{L}^2(\mathbb{R}^2)$ satisfy $\varphi = \mathbb{P}_0(\Delta_E(\delta))\varphi$ with $\|\varphi\| = 1$. Then for all $\varepsilon > 0$ there exists a constant $\tilde{b} > 0$, depending only on n, δ and ε such that we have*

$$\int_{\mathbb{R}^2} \chi_{I_\varepsilon}(x)|\varphi(x,y)|^2 \mathrm{d}x\mathrm{d}y \geqslant 1 - \sqrt{2}\mathrm{e}^{-b^\varepsilon},$$

for $b \geqslant \tilde{b}$. Here χ_{I_ε} is the characteristic function of the interval $I_\varepsilon := [-b^{-1/2+\varepsilon}, b^{-1/2+\varepsilon}]$.

Proof. 1. Due to (2.14) we have

$$\max\{\sup \omega_j^{-1}(\Delta_E(\delta)),\ j = 1, \dots, 2n\} \leqslant \alpha_n b^{1/2},$$

for some constant $\alpha_n > 0$, depending only on n and δ. Hence there is a constant $\beta_n > 0$, depending only on n and δ, such that the estimate

$$Q_j(x,k) := V_{eff}(x,k) - \omega_j(k) \geqslant b^2(|x| - x_n)^2 > 0, \tag{4.4}$$

holds for all $j = 1, \dots, 2n$, $k \in \omega_j^{-1}(\Delta_E(\delta))$ and $|x| \geqslant x_n := \beta_n b^{1/2}$.
2. We will prove that an eigenfunction $\psi_j(k)$, for $k \in \omega_j^{-1}(\Delta_E(\delta))$, decays in the region $|x| \geqslant x_n$. In particular, we will establish for $j = 1, \dots, 2n$ that

$$|\psi_j(x,k)| \leqslant \left(\frac{2b}{\pi}\right)^{1/4} \mathrm{e}^{-b(|x|-x_n)^2/2},\ |x| \geqslant x_n,\ k \in \omega_j^{-1}(\Delta_E(\delta)). \tag{4.5}$$

Let $j \in \mathbb{N}_{2n}^*$ and $k \in \omega_j^{-1}(\Delta_E(\delta))$ be fixed. In light of (4.4) and the differential equation $\psi_j(x,k)'' = Q_j(x,k)\psi_j(x,k)$ we have $\psi_j(x,k)\psi_j'(x,k) < 0$ for $|x| > x_n$, by [15][Proposition 3.1]. This implies that

$$\frac{\psi_j'(x,k)}{\psi_j(x,k)} = \frac{\psi_j'(x,k)\psi_j(x,k)}{\psi_j(x,k)^2} < 0,\ x > x_n. \tag{4.6}$$

Following [15][Lemma 3.5], differentiating $I(x,k) := \psi_j'(x,k)^2 - Q_j(x,k)\psi_j(x,k)^2$, one finds that $\partial_x I(x,k) < 0$ since $Q_j'(x,k) > 0$ in the region $x > x_n$. Since $I(x,k)$ vanishes at infinity, due to the vanishing of $\psi_j(x,k)$ and $\psi_j'(x,k)$ established by [15][Lemma 3.3], this means that $I(x,k) > 0$ in the region $x > x_n$. From this we conclude that

$$\psi_j'(x,k)^2 \geqslant Q_j(x,k)\psi_j(x,k)^2,\ x > x_n. \tag{4.7}$$

As a consequence of (4.4) and (4.6)-(4.7), we find that

$$\frac{\psi_j'(x,k)}{\psi_j(x,k)} \leqslant -\sqrt{Q_j(x,k)} \leqslant -b(x - x_n),\ \text{for } x > x_n.$$

Result (4.5) follows from integrating this differential inequality over the region $x > x_n$ and arguing in the same way as above for $x < -x_n$.

3. Choose b so large that $b^\varepsilon > (1+\beta_n^{1/2})^2$. Then we have $b^{-1/2+\varepsilon} > x_n + b^{(-1+\varepsilon)/2}$ by elementary computations, whence

$$\int_{\mathbb{R}\setminus I_\varepsilon} \psi_j(x,k)^2 dx \leqslant 2\left(\frac{2b}{\pi}\right)^{1/2} \int_{b^{-1/2+\varepsilon}}^{+\infty} \mathrm{e}^{-b(x-x_n)^2} dx \leqslant \sqrt{2} e^{-b^\varepsilon}, \tag{4.8}$$

from (4.5). Finally, since

$$\begin{aligned} \int_{\mathbb{R}^2} \chi_{I_\varepsilon}(x)|\varphi(x,y)|^2 \mathrm{d}x\mathrm{d}y &= \int_{\mathbb{R}^2} \chi_{I_\varepsilon}(x)|\hat{\varphi}(x,k)|^2 \mathrm{d}x\mathrm{d}k \\ &= \sum_{j=1}^{2n} \int_{\omega_j^{-1}(\Delta_E(\delta))} |\beta_j(k)|^2 \left(\int_{I_\varepsilon} \psi_j(x,k)^2 \mathrm{d}x \right) \mathrm{d}k, \end{aligned}$$

by Lemma 3.1, where $\beta_j(k) := \langle \hat{\varphi}(\cdot,k), \psi_j(\cdot,k)\rangle$, the result follows readily from (4.8) and the identity $\sum_{j=1}^{2n} \int_{\omega_j^{-1}(\Delta_E(\delta))} |\beta_j(k)|^2 \mathrm{d}k = 1$. □

4.3. **Persistence of edge currents in time: Asymptotic velocity.** We investigate the time evolution of the edge current under the unitary evolution groups generated by the Iwatsuka Hamiltonians H_0 (1.1), and by the perturbed Iwatsuka Hamiltonians $H(a,q)$ (3.12). This operator generates the unitary time evolution group $e^{-itH(a,q)}$. Let $v_{y,a,q} = (i/2)[H(a,q), M]$, with $M = -y$, be the y-component of the velocity operator. We are interested in evaluating the asymptotic time behavior of $\langle e^{-itH(a,q)}\varphi, v_{y,a,q} e^{-itH(a,q)}\varphi\rangle$ as $t \to \pm\infty$ for appropriate functions φ.

The lower bounds on the edge currents for the unperturbed and the perturbed Iwatsuka models are valid for all time. It we replace $v_y = v_{y,0,0}$ in the expression $J_y(\varphi) = J_{y,0,0}(\varphi) = \langle \varphi, v_y \varphi\rangle$ in Corollary 3.1 by $v_y(t) = v_{y,0,0}(t) := e^{itH_0} v_y e^{-itH_0}$, then the lower bound (3.11) remains valid since the state $\varphi(t) := e^{-itH_0}\varphi$ satisfies $\varphi(t) \in \mathbb{P}_0(\Delta_E(\delta_0))\mathrm{L}^2(\mathbb{R}^2)$ for all time. Similarly, if we replace $v_{y,a,q}$ in (4.2) by its time evolved current $v_{y,a,q}(t)$ using the evolution operator $e^{-itH(a,q)}$, then the lower bound in (4.3) remains valid for all time.

Perturbed Hamiltonians $H(a,q)$ were treated in section 3. Part 2 of Theorem 4.1 states that if the L^∞-norms of a_j, ∇a_j, for $j=1,2$, and of q are small relative to $b_-^{1/2}$ in the sense that condition (3.14) is satisfied, then the edge current $J_{y,a,q}(\psi)$ is bounded from below for all $\psi \in \mathbb{P}_{a,q}(\Delta_E(\delta))\mathrm{L}^2(\mathbb{R}^2)$, where $\Delta_E(\delta)$ is defined at the beginning of section 3 and (E,δ) are as in Proposition 3.1 and Proposition 3.2. The boundedness of a_j, ∇a_j, and of q is rather restrictive. From the form of the current operator in (4.1), it would appear that only $\|a_2\|_\infty$ needs to be controlled. We prove here that if we limit the support of the perturbation (a_1,a_2,q) to a strip of arbitrary width R in the y-direction, and require only that $\|a_2\|_\infty$ be small relative to $b_-^{1/2}$, then the *asymptotic velocity* associated with energy intervals $\Delta_E(\delta)$ and the perturbed Hamiltonian $H(a,q)$ exists and satisfies the same lower bound as in (4.3). Furthermore, the spectrum in $\Delta_E(\delta)$ is absolutely continuous. This means that the edge current is stable with respect to a different class of perturbations than in Theorem 4.1.

We recall that the *asymptotic velocity* associated with a pair of self-adjoint operators (H_0, H_1) is defined in terms of the local wave operators for the pair, see, for example [9, section 4.5–4.6]. The local wave operators $\Omega_\pm(\Delta)$ for an energy interval $\Delta \subset \mathbb{R}$ are defined as the strong limits:

$$\Omega_\pm(\Delta) := s - \lim_{t\to\pm\infty} e^{itH_1} e^{-itH_0} \mathbb{P}_{0,ac}(\Delta), \tag{4.9}$$

where $\mathbb{P}_{0,ac}(\Delta)$ is the spectral projector for the absolutely continuous subspace of H_0 associated with the interval Δ. For any φ, we define the *asymptotic velocity* $V_y^\pm(\Delta)$ of the state φ by

$$\langle \varphi, V_y^\pm(\Delta)\varphi\rangle := \langle \varphi, \Omega_\pm(\Delta) v_y \Omega_\pm(\Delta)^* \varphi\rangle. \tag{4.10}$$

In the case that H_0 commutes with v_y, it is easily seen from the definition (4.9) that

$$\langle \varphi, V_y^\pm(\Delta)\varphi\rangle = \lim_{t\to\pm\infty} \langle \varphi, e^{itH_1} \mathbb{P}_{0,ac}(\Delta) v_y \mathbb{P}_{0,ac}(\Delta) e^{-itH_1}\varphi\rangle.$$

Our main result is the existence of the asymptotic velocity (4.10) in the y-direction for the perturbed operators $H(a, q)$ described in section 3.3. We prove that the asymptotic velocity satisfies the lower bound given in (4.11) provided the perturbations (a, q) have compact support in the y-direction. The local wave operators appearing in the definition (4.10) are constructed from the pair (H_0, H_1) where H_0 is the unperturbed Iwatsuka Hamiltonian and $H_1 = H(a, q)$. As discussed in section 2.5, the spectrum of H_0 is purely absolutely continuous.

Theorem 4.2. *Let b, n, E, and δ_0, be as in Proposition 3.1 and for any $0 < \delta \leqslant \delta_0$, let $\Delta_E(\delta)$ be as defined in section 3.1. Suppose the perturbation (a_1, a_2, q) and $0 < \delta < \delta_0$ satisfy the hypotheses of Theorem 4.1. Furthermore, suppose that (a, q) have their support in the set $\{(x, y) \mid |y| < R\}$, for some $0 < R < \infty$. Then for any $\varphi \in \operatorname{Ran} \mathbb{P}_{a,q}(\Delta_E(\delta))$, we have*

$$\langle \varphi, V_y^\pm(\Delta)\varphi\rangle \geqslant \frac{c_n}{4} b^{1/2} \|\varphi\|^2. \tag{4.11}$$

The proof of Theorem 4.2 closely follows the proof in [14, section 7] (see also [13, section 4]). We mention the main points. We first prove the existence of the local wave operators (4.9) for the pair H_0 and $H_1 = H(a, q)$, and the interval $\Delta_E(\delta)$, as in the theorem. The key point is that in the application of the method of stationary phase, we use the positivity bound (3.3). We then use the intertwining properties of the local wave operators to find

$$\begin{aligned}
\langle \varphi, V_y^\pm(\Delta_E(\delta))\varphi\rangle &= \langle \varphi, \Omega_\pm(\Delta_E(\delta)) v_{y,a,q} \Omega_\pm(\Delta_E(\delta))^* \varphi\rangle \\
&= \langle \Omega_\pm(\Delta_E(\delta))^* \mathbb{P}_{a,q}(\Delta_E(\delta))\varphi, v_{y,a,q} \Omega_\pm(\Delta_E(\delta))^* \mathbb{P}_{a,q}(\Delta_E(\delta))\varphi\rangle \\
&= \langle \mathbb{P}_0(\Delta_E(\delta))\Omega_\pm(\Delta_E(\delta))^*\varphi, v_{y,a,q} \mathbb{P}_0(\Delta_E(\delta))\Omega_\pm(\Delta_E(\delta))^*\varphi\rangle \\
&\geqslant \frac{c_n}{4} b^{1/2} \|\mathbb{P}_0(\Delta_E(\delta))\Omega_\pm(\Delta_E(\delta))^*\varphi\|^2,
\end{aligned}$$

where we used the lower bound in part 2 of Theorem 4.1. To complete the proof, we again use the intertwining relation to write

$$\|\mathbb{P}_0(\Delta_E(\delta))\Omega_\pm(\Delta_E(\delta))^*\varphi\| = \|\Omega_\pm(\Delta_E(\delta))^* \mathbb{P}_{a,q}(\Delta_E(\delta))\varphi\| = \|\varphi\|,$$

since the local wave operators are partial isometries.

5. Asymptotic behavior of the eigenvalue counting function for negative perturbations of H_0 below $\inf \sigma_{\mathrm{ess}}(H_0)$

The infimum of the spectrum of the unperturbed operator H_0 is given by the unique minimum of the first band function $\omega_1(k)$ at $k = \kappa_1$. This value is below the first Landau level b which is the asymptotic limit of the first band function $b = \lim_{k\to\infty} \omega_1(k)$. Because the infimum is achieved at a finite value κ_1, the perturbation of H_0 by a negative potential may create a sequence of eigenvalues accumulating at $\omega_1(\kappa_1)$ from below. In this section, we apply the method introduced in [18] to describe the discrete spectrum of the perturbed operator $H := H_0 - V$ near the infimum of its essential spectrum, when the scalar potential $V = V(x, y) \geqslant 0$ decays suitably as $|y| \to \infty$. For potentials of this type, we prove that there are an infinite number of eigenvalues accumulating at $\mathcal{E}_1 = \inf \sigma_{\mathrm{ess}}(H_0) = \inf \sigma_{\mathrm{ess}}(H)$ from below. We also describe the behavior of the eigenvalue counting function.

The only information on H_0 we use here is the local behavior of the first band function $\omega_1(k)$ at its unique minimum $k = \kappa_1$. In addition to the existence of the unique minimum at κ_1, it is crucial that the minimum is non-degenerate, which is to say that the effective mass is positive. This is the content of Proposition 2.4. From this result and the analyticity of $k \mapsto \omega_1(k)$, it follows that the first band function satisfies the asymptotic identity

$$\omega_1(k) - \mathcal{E}_1 = \beta_1(k - \kappa_1)^2 + O((k - \kappa_1)^3),\ k \to \kappa_1,$$

with $\beta_1 := \omega_1''(\kappa_1)/2 > 0$.

5.1. **Statement of the result.** We first introduce the following notation. Let H be a linear self-adjoint operator acting in a given separable Hilbert space. Assume that $\mathcal{E} = \inf \sigma_{\mathrm{ess}}(H) > -\infty$. The eigenvalue counting function $N(\mu; H)$, $\mu \in (-\infty, \mathcal{E})$, denotes the number of the eigenvalues of H lying on the interval $(-\infty, \mu)$, and counted with the multiplicities. We recall that $\psi_1(x, k)$ is the first eigenfunction of the fiber operator $h(k)$ with band function $\omega_1(k)$.

Theorem 5.1. *Let $V(x, y) \in \mathrm{L}^\infty(\mathbb{R}^2)$ satisfy the following two conditions:*

i.) $\exists(\alpha, C) \in (0, 2) \times \mathbb{R}_+^*$ *so that*

$$0 \leqslant V(x, y) \leqslant C(1 + |x|)^{-\alpha}(1 + |y|)^{-\alpha},\ x, y \in \mathbb{R};$$

ii.) $\exists L > 0$ *so that* $\lim_{|y|\to\infty} |y|^\alpha \int_{\mathbb{R}} V(x, y)\psi_1(x, \kappa_1)^2 dx = L$.

Then we have

$$\lim_{\lambda \downarrow 0} \lambda^{-\frac{1}{2}+\frac{1}{\alpha}} N(\mathcal{E}_1 - \lambda; H_0 - V) = \frac{2}{\alpha\pi}\beta_1^{-1/2} L^{1/\alpha} \mathrm{B}\left(\frac{3}{2}, \frac{1}{\alpha} - \frac{1}{2}\right), \tag{5.1}$$

where $\mathrm{B}(\cdot, \cdot)$ *is the Euler beta function* [2, section 6.2] *and* $\beta_1 := \omega_1''(\kappa_1)/2 > 0$ *is the effective mass.*

5.2. **Notations and auxiliary results.** This subsection presents some notation and several auxiliary results needed for the proof of Theorem 5.1, which is presented in §5.3.

For a linear compact self-adjoint operator H acting in a separable Hilbert space, we define

$$n(s; H) := \text{rank}\ \mathbb{P}_{(s,\infty)}(H),\ s > 0,$$

where $\mathbb{P}_I(H)$ denotes the spectral projection of H associated with the interval $I \subset \mathbb{R}$. Let X_1 and X_2 be two separable Hilbert spaces. For a linear compact operator $H : X_1 \to X_2$, we set

$$\mathfrak{n}(s; H) := n(s^2; H^*H),\ s > 0. \tag{5.2}$$

If $H_j : X_1 \to X_2$, $j = 1, 2$, are two linear compact operators, we will use Ky Fan inequality

$$\mathfrak{n}(s_1 + s_2, H_1 + H_2) \leqslant \mathfrak{n}(s_1, H_1) + \mathfrak{n}(s_2, H_2), \tag{5.3}$$

which holds for $s_1 > 0$ and $s_2 > 0$ according to [3, Chapter I, Eq. (1.31)] and [11, Chapter II, Section 2, Corollary 2.2].

For further reference, we recall from [18, Eq. (2.1) & Lemma 2.3] the following technical result.

Lemma 5.1. [18, Lemma 2.3] *Let $G : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}^2)$ be a bounded operator with integral kernel $g \in \mathrm{L}^\infty(\mathbb{R}^3)$. Then for every $f \in \mathrm{L}^r(\mathbb{R}^2)$ and $h \in \mathrm{L}^r(\mathbb{R})$ with $r \in [2, \infty)$, we have*

$$\mathfrak{n}(s; fGh) \leqslant C_r(G) s^{-r} \|f\|^r_{\mathrm{L}^r(\mathbb{R}^2)} \|h\|^r_{\mathrm{L}^r(\mathbb{R})},\ s > 0,$$

where $C_r(G) := \|g\|^{4/r}_{\mathrm{L}^\infty(\mathbb{R}^3)} \|G\|^{2(r-2)/r}$.

For $\delta > 0$ fixed, let $\chi = \chi_\delta$ denote the characteristic function of the interval $I = I_\delta := (\kappa_1 - \delta, \kappa_1 + \delta)$. We shall actually apply Lemma 5.1 in §5.3 with $G = \Gamma_j$, $j = 0, 1$, where $\Gamma_j : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}^2)$ is the integral operator with kernel

$$\gamma_0(x, y, k) := \frac{1}{\sqrt{2\pi}} \psi_1(x, \kappa_1) e^{-iyk} \chi(k),\ (x, y) \in \mathbb{R}^2,\ k \in \mathbb{R}, \tag{5.4}$$

and

$$\gamma_1(x, y, k) := \frac{1}{\sqrt{2\pi}} \left(\frac{\psi_1(x, k) - \psi_1(x, \kappa_1)}{k - \kappa_1} \right) e^{-iyk} \chi(k),\ (x, y) \in \mathbb{R}^2,\ k \in \mathbb{R} \backslash \{\kappa_1\}. \tag{5.5}$$

Lemma 5.2. *We have $\gamma_j \in \mathrm{L}^\infty(\mathbb{R}^3_{x,y,k})$ for $j = 0, 1$.*

Proof. In view of (5.4)-(5.5), it suffices to prove that $(x, k) \mapsto \psi_1(x, k)$ and $(x, k) \mapsto (\psi_1(x, k) - \psi_1(x, \kappa_1))/(k - \kappa_1)$ are respectively bounded in $\mathbb{R} \times I$ and $\mathbb{R} \times (I \setminus \{\kappa_1\})$. The eigenfunction $\psi_1(\cdot, k)$ is a solution to the second order ordinary differential equation

$$-\varphi''(x) + W(x, k)\varphi(x) = 0,\ x \in \mathbb{R}, \tag{5.6}$$

where $W(x, k) := (b|x| - k)^2 - \omega_1(k)$. The potential $W(x, k)$ is greater than δ^2 provided $|x| > x_1 := (b^{1/2} + \kappa_1 + 2\delta)/b$, uniformly in $k \in I$. It follows from [13, Lemma B.3] that

$$0 < \psi_1(x, k) \leqslant \psi_1(\pm x_1, k) e^{-\delta(|x| - x_1)},\ x \geqslant x_1,\ k \in I.$$

Since $(x, k) \mapsto \psi_1(x, k)$ is continuous in $\mathbb{R} \times I$, this implies the result for $j = 0$.

Next, bearing in mind that the $\mathrm{L}^2(\mathbb{R})$-valued function $k \mapsto \psi_1(\cdot,k)$ is real analytic, we deduce from (5.6) that $\Phi(\cdot,k) := \partial_k \psi_1(\cdot,k)$ is solution to the equation

$$-\varphi''(x) + W(x,k)\varphi(x) = -F(x,k),\ x \in \mathbb{R},$$

where $F(x,k) := \partial_k W(x,k)\psi_1(x,k) = (2(k-b|x|) - \omega_1'(k))\psi_1(x,k)$. Therefore we get that

$$\begin{aligned} & \|\Phi'(\cdot,k)\|^2_{\mathrm{L}^2(\mathbb{R})} + \|(b|x|-k)\Phi(\cdot,k)\|^2_{\mathrm{L}^2(\mathbb{R})} \\ \leqslant\ & (C+\omega_1(k))\,\|\Phi(\cdot,k)\|_{\mathrm{L}^2(\mathbb{R})})\|\Phi(\cdot,k)\|_{\mathrm{L}^2(\mathbb{R})}, \end{aligned} \tag{5.7}$$

with $C := \sup_{k\in I} \|F(\cdot,k)\|_{\mathrm{L}^2(\mathbb{R})} < \infty$, by standard computations. Since $\sup_{k\in I} \|\Phi(\cdot,k)\|_{\mathrm{L}^2(\mathbb{R})} < \infty$, (5.7) thus entails that $\sup_{k\in I} \|\Phi(\cdot,k)\|_{\mathrm{H}^1(\mathbb{R})} < \infty$. From this and the estimate

$$\Phi(x,k)^2 = 2\int_{-\infty}^{x} \Phi(\tilde{x},k)\Phi'(\tilde{x},k)d\tilde{x} \leqslant \|\Phi(\cdot,k)\|^2_{\mathrm{H}^1(\mathbb{R})},\ x\in\mathbb{R},\ k\in\mathbb{R},$$

then follows that $\sup_{(x,k)\in\mathbb{R}\times I} |\Phi(x,k)| < \infty$. This yields the result for $j=1$ and terminates the proof. □

Finally, since the proof of Theorem 5.1 is obtained by expressing $\lim_{\lambda\downarrow 0} N(\mathcal{E}_1-\lambda; H_0-V)$ in terms of the asymptotics of the eigenvalue counting function for the discrete spectrum of a second-order ordinary differential operators on the real line, we recall from [5, Lemma 4.9] the following

Lemma 5.3. *Assume that $\mathcal{Q} = \overline{\mathcal{Q}} \in \mathrm{L}^\infty(\mathbb{R})$ satisfies the two following conditions:*

i.) $\exists(\alpha, C) \in (0,2)\times\mathbb{R}_+^*$ *so that* $|\mathcal{Q}(x)| \leqslant C(1+|x|)^{-\alpha},\ x\in\mathbb{R}$;
ii.) $\exists \ell > 0$ *so that* $\lim_{|x|\to\infty} |x|^\alpha \mathcal{Q}(x) = \ell$.

For any $\mathfrak{m} > 0$, let $\mathcal{H}(\mathfrak{m},\mathcal{Q}) := -\mathfrak{m}^2\frac{d^2}{dx^2} - \mathcal{Q}$ be the 1D Schrödinger operator with domain $\mathrm{H}^2(\mathbb{R})$, self-adjoint in $\mathrm{L}^2(\mathbb{R})$.
Then we have

$$\lim_{\lambda\downarrow 0} \lambda^{\frac{1}{\alpha}-\frac{1}{2}} N(-\lambda;\mathcal{H}(\mathfrak{m},\mathcal{Q})) = \frac{2\ell^{\frac{1}{\alpha}}}{\pi\alpha\mathfrak{m}} B\left(\frac{3}{2},\frac{1}{\alpha}-\frac{1}{2}\right).$$

The proof of Lemma 5.3, which is similar to the one of [19, Theorem XIII.82], can be found in [16].

5.3. **Proof of Theorem 5.1.** The proof consists of four parts.

5.3.1. *Part I: Projection on the bottom of the first band function.* We define $\mathcal{V} := \mathcal{F}V\mathcal{F}^*$ and recall that $\mathcal{H}_0 = \mathcal{F}H_0\mathcal{F}^*$. The first part of the proof is to show that the asymptotics of $N(\mathcal{E}_1-\lambda;H_0-V)$ as $\lambda\downarrow 0$ is determined by the asymptotics of the eigenvalue counting function for a reduced operator obtained from the projection of the operator $\mathcal{H}_0-\mathcal{V}$ to the bottom of the first band function. First of all, we remark that the multiplier by V is H_0-compact since $V(x,y)$ goes to zero as $|(x,y)|$ tends to infinity. As a consequence we have

$$\inf\sigma_{\mathrm{ess}}(H_0-V) = \inf\sigma_{\mathrm{ess}}(H_0),$$

hence $N(\mathcal{E}_1-\lambda; H_0-V)<\infty$ for any $\lambda>0$. Furthermore, since $V\in \mathrm{L}^\infty(\mathbb{R}^2)$, the operator H_0-V is lower semibounded. The operator H_0-V is unitarily equivalent to $\mathcal{H}_0-\mathcal{V}$, so we have

$$N(\mathcal{E}_1-\lambda; H_0-V)=N(\mathcal{E}_1-\lambda;\mathcal{H}_0-\mathcal{V}),\ \lambda>0.$$

Let $P:\mathrm{L}^2(\mathbb{R}^2)\to\mathrm{L}^2(\mathbb{R}^2)$ be the orthogonal projection defined by

$$(Pu)(x,k):=\left(\int_{\mathbb{R}} u(t,k)\psi_1(t,k)dt\right)\chi(k)\psi_1(x,k),\ (x,k)\in\mathbb{R}^2, \tag{5.8}$$

where we recall that χ denotes the characteristic function of the interval $I:=(\kappa_1-\delta,\kappa_1+\delta)$ for some fixed $\delta>0$.

Lemma 5.4. *Let $\mathcal{H}_1(t)$, $t\in\mathbb{R}$, be the operator $P(\mathcal{H}_0-(1+t)\mathcal{V})P$ with domain $P\,\mathrm{Dom}(\mathcal{H}_0)$. Then there is a constant $N_0\geqslant 0$, independent of λ, such that we have*

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_1(0))\leqslant N(\mathcal{E}_1-\lambda;\mathcal{H}_0-\mathcal{V})\leqslant N(\mathcal{E}_1-\lambda;\mathcal{H}_1(0))+N_0,\ \lambda>0. \tag{5.9}$$

Proof. Set $Q:=I-P$. For all $u\in\mathrm{L}^2(\mathbb{R}^2)$ and $\varepsilon>0$ it holds true that

$$\begin{aligned}|\langle(P\mathcal{V}Q+Q\mathcal{V}P)u,u\rangle| &= 2\left|\mathrm{Re}\left(\langle\mathcal{V}^{1/2}Pu,\mathcal{V}^{1/2}Qu\rangle\right)\right|\\ &\leqslant\ 2\|\mathcal{V}^{1/2}Pu\|\|\mathcal{V}^{1/2}Qu\|\leqslant\varepsilon\langle P\mathcal{V}Pu,u\rangle+\varepsilon^{-1}\langle Q\mathcal{V}Qu,u\rangle,\end{aligned}$$

which entails

$$-\varepsilon P\mathcal{V}P-\varepsilon^{-1}Q\mathcal{V}Q\leqslant P\mathcal{V}Q+Q\mathcal{V}P\leqslant\varepsilon P\mathcal{V}P+\varepsilon^{-1}Q\mathcal{V}Q,$$

in the sense of quadratic forms. From this and the elementary identity $\mathcal{H}_0=P\mathcal{H}_0P+Q\mathcal{H}_0Q$ then follows that

$$\mathcal{H}_1(\varepsilon)\oplus\mathcal{H}_2(\varepsilon)\leqslant\mathcal{H}_0-\mathcal{V}\leqslant\mathcal{H}_1(-\varepsilon)\oplus\mathcal{H}_2(-\varepsilon),\ \varepsilon>0, \tag{5.10}$$

where $\mathcal{H}_2(t)$, $t\in\mathbb{R}^*$, is the operator $Q(\mathcal{H}_0-(1+t^{-1})\mathcal{V})Q$ with domain $Q\,\mathrm{Dom}(\mathcal{H}_0)$, and the symbol $\oplus$ indicates an orthogonal sum. Therefore, for every $\lambda>0$ and $\varepsilon>0$ fixed, the left inequality in (5.10) implies

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_0-\mathcal{V})\leqslant N(\mathcal{E}_1-\lambda;\mathcal{H}_1(\varepsilon))+N(\mathcal{E}_1-\lambda;\mathcal{H}_2(\varepsilon)), \tag{5.11}$$

while the right one yields

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_0-\mathcal{V})\geqslant N(\mathcal{E}_1-\lambda;\mathcal{H}_1(-\varepsilon))+N(\mathcal{E}_1-\lambda;\mathcal{H}_2(-\varepsilon))\geqslant N(\mathcal{E}_1-\lambda;\mathcal{H}_1(-\varepsilon)). \tag{5.12}$$

Further, the multiplier by $\mathcal{V}$ being $\mathcal{H}_0$-compact, $Q\mathcal{V}Q$ is $Q\mathcal{H}_0$-compact and

$$\sigma_{\mathrm{ess}}(\mathcal{H}_2(\varepsilon))=\sigma_{\mathrm{ess}}(Q\mathcal{H}_0),\ \varepsilon>0. \tag{5.13}$$

On the other hand we have $\inf\sigma_{\mathrm{ess}}(Q\mathcal{H}_0)=\min\{\omega_1(\kappa_1+\delta),\min_{k\in\mathbb{R}}\omega_2(k)\}>\mathcal{E}_1$ hence

$$N_0:=N(\mathcal{E}_1;Q\mathcal{H}_0)<\infty,$$

and (5.13) yields

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_2(\varepsilon))=N(\mathcal{E}_1-\lambda;Q\mathcal{H}_0)\leqslant N_0,\ \lambda>0,\ \varepsilon>0. \tag{5.14}$$

Putting (5.11) and (5.14) together, we get that

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_0-\mathcal{V})\leqslant N(\mathcal{E}_1-\lambda;\mathcal{H}_1(\varepsilon))+N_0,\ \lambda>0,\ \varepsilon>0. \tag{5.15}$$

Letting $\varepsilon\downarrow 0$ in (5.12) and (5.15), we obtain (5.9). ∎

5.3.2. *Part II: Singular integral operator decomposition.* This part involves relating the number of eigenvalues accumulating below the bottom of the essential spectrum of $\mathcal{H}_1(0)$, to the local behavior of $\omega_1(k)$ and $\psi_1(\cdot,k)$ at κ_1.

The main tool we use for this is the Birman-Schwinger principle, which, in this situation, implies

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_1(0)) = n(1; P(\mathcal{H}_0-\mathcal{E}_1+\lambda)^{-1/2}\mathcal{V}(\mathcal{H}_0-\mathcal{E}_1+\lambda)^{-1/2}P). \quad (5.16)$$

In view of (5.8) and (5.16), we set

$$a(k,\lambda) := (\omega_1(k)-\mathcal{E}_1+\lambda)^{-1/2},\ k\in\mathbb{R},\ \lambda>0, \quad (5.17)$$

and denote by $\Gamma : \mathrm{L}^2(\mathbb{R}_k)\to \mathrm{L}^2(\mathbb{R}^2_{x,y})$ the operator with integral kernel

$$\gamma(x,y,k) := \frac{1}{\sqrt{2\pi}}\psi_1(x,k)\mathrm{e}^{iyk}\chi(k),\ (x,y)\in\mathbb{R}^2,\ k\in\mathbb{R}.$$

For every $\lambda>0$ the operator $\chi a(\lambda)\Gamma^* V\Gamma a(\lambda)\chi$, where the symbol $a(\lambda)$ (resp. χ) stands for the multiplier by $a(\cdot,\lambda)$ (resp. $\chi(\cdot)$) in $\mathrm{L}^2(\mathbb{R}_k)$, is self-adjoint and nonnegative in $\mathrm{L}^2(\mathbb{R}_k)$. Furthermore we get

$$P(\mathcal{H}_0-\mathcal{E}_1+\lambda)^{-1/2}\mathcal{V}(\mathcal{H}_0-\mathcal{E}_1+\lambda)^{-1/2}P = \mathcal{U}^*\chi a(\lambda)\Gamma^* V\Gamma a(\lambda)\chi\mathcal{U}, \quad (5.18)$$

by direct calculation, where $\mathcal{U} : \operatorname{Ran} P\to \mathrm{L}^2(I)$ is the unitary transform

$$(\mathcal{U}f)(k) := \left(\int_{\mathbb{R}} f(x,k)\psi_1(x,k)dx\right)\chi(k),\ k\in\mathbb{R}.$$

From (5.16)-(5.18) then follows that

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_1(0)) = n(1;\chi a(\lambda)\Gamma^* V\Gamma a(\lambda)\chi),\ \lambda>0. \quad (5.19)$$

Putting $W := V^{1/2}$ we deduce from (5.2) and (5.19) that

$$N(\mathcal{E}_1-\lambda;\mathcal{H}_1(0)) = \mathfrak{n}(1; W\Gamma a(\lambda)\chi),\ \lambda>0. \quad (5.20)$$

5.3.3. *Part III: Reduction to the quadratic leading term of the first band function.* Due to (5.9) and (5.20), we are left with the task of computing the asymptotics of $\mathfrak{n}(1;W\Gamma a(\lambda)\chi)$ as $\lambda\downarrow 0$. In this subsection, we shall prove that Γ and $a(\lambda)\chi$ may be replaced by, respectively, Γ_0 and $\mathfrak{a}(\lambda)\chi$, in the above expression. The operator $\Gamma_0 : \mathrm{L}^2(\mathbb{R})\to\mathrm{L}^2(\mathbb{R}^2)$ is the operator with integral kernel $\gamma_0(x,y,k)$ given by (5.4). We obtain $\mathfrak{a}(\cdot,\lambda)$ from $a(\cdot,\lambda)$ in (5.17) by replacing $\omega_1(k)$ by the first two terms of the expansion of $\omega_1(k)$ about κ_1:

$$\mathfrak{a}(k,\lambda) := \left(\beta_1(k-\kappa_1)^2+\lambda\right)^{-1/2},\ k\in\mathbb{R},\ \lambda>0. \quad (5.21)$$

Lemma 5.5. *Let $r\geqslant 2$ fulfill $r>2/\alpha$. Then there exists a constant $N_r\geqslant 0$ such that the estimates*

$$\begin{aligned}\mathfrak{n}((1+\varepsilon)^3; W\Gamma_0\mathfrak{a}(\lambda)) - N_r\varepsilon^{-r} &\leqslant \mathfrak{n}(1; W\Gamma a(\lambda)\chi)\\ &\leqslant \mathfrak{n}((1-\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)) + N_r\varepsilon^{-r}, \quad (5.22)\end{aligned}$$

hold for all $\lambda>0$ and $\varepsilon\in(0,1)$.

Proof. 1. We use the decomposition $\Gamma a(\lambda)\chi = \sum_{j=0}^{1} \Gamma_j a_j(\lambda)\chi$, where $\Gamma_1 : \mathrm{L}^2(\mathbb{R}) \to \mathrm{L}^2(\mathbb{R}^2)$ is the operator with integral kernel $\gamma_1(x,y,k)$ defined in (5.5), and

$$a_j(k,\lambda) := (k-\kappa_1)^j a(k,\lambda),\ j=0,1.$$

Since $\gamma_j \in \mathrm{L}^\infty(\mathbb{R}^3)$, $j=0,1$, by Lemma 5.2, the operators Γ_j are bounded with

$$\|\Gamma_0\| = 1 \text{ and } \|\Gamma_1\|^2 \leqslant \sup_{k\in I} \int_{\mathbb{R}} \left(\frac{\psi_1(x,k)-\psi_1(x,\kappa_1)}{k-\kappa_1} \right)^2 dx.$$

We notice from (5.3) that

$$\begin{aligned} & \mathfrak{n}(1+\varepsilon; W\Gamma_0 a_0(\lambda)\chi) - \mathfrak{n}(\varepsilon; W\Gamma_1 a_1(\lambda)\chi) \\ \leqslant\ & \mathfrak{n}(1; W\Gamma a(\lambda)\chi) \\ \leqslant\ & \mathfrak{n}(1-\varepsilon; W\Gamma_0 a_0(\lambda)\chi) + \mathfrak{n}(\varepsilon; W\Gamma_1 a_1(\lambda)\chi),\ \lambda>0,\ \varepsilon\in(0,1). \end{aligned} \tag{5.23}$$

2. We obtain an upper bound for $\mathfrak{n}(\varepsilon; W\Gamma_1 a_1(\lambda)\chi)$ in (5.23) from Lemma 5.1 taking $G=\Gamma_1$, $f=W$, and $h=a_1(\cdot,\lambda)\chi$. We get that

$$\begin{aligned} \mathfrak{n}(\varepsilon; W\Gamma_1 a_1(\lambda)\chi) &\leqslant C_r(\Gamma_1)\varepsilon^{-r}\|W\|^r_{\mathrm{L}^r(\mathbb{R}^2)} \|a_1(\cdot,\lambda)\chi\|^r_{\mathrm{L}^r(\mathbb{R})} \\ &\leqslant n_r \varepsilon^{-r},\ \lambda>0,\ \varepsilon\in(0,1), \end{aligned}$$

with $n_r := C_r(\Gamma_1)\|W\|^r_{\mathrm{L}^r(\mathbb{R}^2)}\|a_1(\cdot,0)\chi\|^r_{\mathrm{L}^r(\mathbb{R})}$. From this and (5.23) then follows that

$$\begin{aligned} \mathfrak{n}(1+\varepsilon; W\Gamma_0 a_0(\lambda)\chi) - n_r\varepsilon^{-r} &\leqslant \mathfrak{n}(1; W\Gamma a(\lambda)\chi) \\ &\leqslant \mathfrak{n}(1-\varepsilon; W\Gamma_0 a_0(\lambda)\chi) + n_r\varepsilon^{-r}, \end{aligned} \tag{5.24}$$

for $\lambda>0,\ \varepsilon\in(0,1)$.

3. Next, recalling that χ is the characteristic function of the interval $(\kappa_1-\delta,\kappa_1+\delta)$, for $\varepsilon\in(0,1)$ fixed, we choose $\delta>0$ so small that

$$(1+\varepsilon)^{-1}\mathfrak{a}(k,\lambda)\chi(k) \leqslant a_0(k,\lambda)\chi(k) \leqslant (1-\varepsilon)^{-1}\mathfrak{a}(k,\lambda)\chi(k),\ k\in\mathbb{R},\ \lambda>0,$$

where $\mathfrak{a}(\lambda)$ is defined in (5.21). It follows from this and the simple identity that $n(s;tH) = n(t^{-1}s;H)$, for $s,t>0$, that we have

$$\begin{aligned} \mathfrak{n}(s(1+\varepsilon); W\Gamma_0\mathfrak{a}(\lambda)\chi) &\leqslant \mathfrak{n}(s; W\Gamma_0 a_0(\lambda)\chi) \\ &\leqslant \mathfrak{n}(s(1-\varepsilon); W\Gamma_0\mathfrak{a}(\lambda)\chi),\ s>0, \end{aligned} \tag{5.25}$$

Moreover (5.3) and the minimax principle yield

$$\begin{aligned} \mathfrak{n}(s(1+\varepsilon); W\Gamma_0\mathfrak{a}(\lambda)) - \mathfrak{n}(s\varepsilon; W\Gamma_0\mathfrak{a}(\lambda)(1-\chi)) &\leqslant \mathfrak{n}(s; W\Gamma_0\mathfrak{a}(\lambda)\chi) \\ &\leqslant \mathfrak{n}(s; W\Gamma_0\mathfrak{a}(\lambda)), \end{aligned} \tag{5.26}$$

for $s>0, \varepsilon\in(0,1)$, as we have $\mathfrak{a}(\lambda)\Gamma_0^* V\Gamma_0\mathfrak{a}(\lambda) \geqslant \chi\mathfrak{a}(\lambda)\Gamma_0^* V\Gamma_0\mathfrak{a}(\lambda)\chi$ in the sense of quadratic forms. Combining the second inequality of (5.25) with $s=1-\varepsilon$ with the second inequality of (5.26) with $s=(1-\varepsilon)^2$, we obtain

$$\begin{aligned} \mathfrak{n}(1-\varepsilon; W\Gamma_0 a_0(\lambda)\chi) &\leqslant \mathfrak{n}((1-\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)\chi) \\ &\leqslant \mathfrak{n}((1-\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)),\ \lambda>0. \end{aligned} \tag{5.27}$$

Similarly, combining the first inequality of (5.26) with $s = (1+\varepsilon)^2$ with the first inequality of (5.25) for $s = 1+\varepsilon$, we find that

$$\begin{aligned} & \mathfrak{n}((1+\varepsilon)^3; W\Gamma_0\mathfrak{a}(\lambda)) - \mathfrak{n}(\varepsilon(1+\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)(1-\chi)) \\ \leqslant\ & \mathfrak{n}((1+\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)\chi) \\ \leqslant\ & \mathfrak{n}(1+\varepsilon; W\Gamma_0 a_0(\lambda)\chi),\ \lambda > 0. \end{aligned} \tag{5.28}$$

4. In order to evaluate $\mathfrak{n}(\varepsilon(1+\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)(1-\chi))$ in (5.28), we use Lemma 5.1 with $G = \Gamma_0$, $f = W$ and $h = \mathfrak{a}(\cdot,\lambda)(1-\chi)$. We obtain that

$$\begin{aligned} & \mathfrak{n}(\varepsilon(1+\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)(1-\chi)) \\ \leqslant\ & C_r(\Gamma_0)\varepsilon^{-r}(1+\varepsilon)^{-2r}\|W\|^r_{\mathrm{L}^r(\mathbb{R}^2)}\|\mathfrak{a}(\cdot,\lambda)(1-\chi)\|^r_{\mathrm{L}^r(\mathbb{R})} \\ \leqslant\ & n'_r\varepsilon^{-r},\ \lambda > 0, \end{aligned} \tag{5.29}$$

with $n'_r := C_r(\Gamma_0)\|W\|^r_{\mathrm{L}^r(\mathbb{R}^2)}\|\mathfrak{a}(\cdot,0)(1-\chi)\|^r_{\mathrm{L}^r(\mathbb{R})}$. Finally, (5.22) follows from (5.24) and (5.27)–(5.29) upon setting $N_r := n_r + n'_r$. □

Summing up (5.9) and (5.20)-(5.22), we have so far derived the following upper bound:

$$\begin{aligned} & \mathfrak{n}((1+\varepsilon)^3; W\Gamma_0\mathfrak{a}(\lambda)) - N_r\varepsilon^{-r} \\ \leqslant\ & N(\mathcal{E}_1 - \lambda; H_0 - V) \\ \leqslant\ & \mathfrak{n}((1-\varepsilon)^2; W\Gamma_0\mathfrak{a}(\lambda)) + N_0 + N_r\varepsilon^{-r},\ \lambda > 0,\ \varepsilon \in (0,1). \end{aligned} \tag{5.30}$$

5.3.4. *Part IV: Reduction to a 1D problem.* Let $\mathfrak{h}(s)$, $s > 0$, be the Hamiltonian $\mathcal{H}(\mathfrak{m},\mathcal{Q})$ introduced in Lemma 5.3, with $\mathfrak{m} := \beta_1^{1/2}$ and $\mathcal{Q} := s^{-2}\int_{\mathbb{R}} V(x,y)\psi_1(x,\kappa_1)^2 dx$. By the Birman-Schwinger principle, we have

$$\mathfrak{n}(s; W\Gamma_0\mathfrak{a}(\lambda)) = n(1; s^{-2}\mathfrak{a}(\lambda)\Gamma_0^* V\Gamma_0\mathfrak{a}(\lambda)) = N(-\lambda; \mathfrak{h}(s)),\ s > 0,\ \lambda > 0.$$

Lemma 5.3 applied to the Hamiltonian $\mathfrak{h}(s)$, $s > 0$, yields the asymptotic

$$\lim_{\lambda\downarrow 0} \lambda^{-\frac{1}{2}+\frac{1}{\alpha}} N(-\lambda; \mathfrak{h}(s)) = c(\alpha,\beta_1,L)s^{-2/\alpha},\ s > 0, \tag{5.31}$$

with $c(\alpha,\beta_1,L) := 2/(\alpha\pi)\beta_1^{-1/2}L^{1/\alpha}\mathrm{B}(3/2, 1/\alpha - 1/2)$. To obtain a lower bound on $N(\mathcal{E}_1 - \lambda; H_0 - V)$ from the first inequality in (5.30), we take $s = (1+\varepsilon)^3$ in (5.31) and obtain

$$\liminf_{\lambda\downarrow 0} \lambda^{-\frac{1}{2}+\frac{1}{\alpha}} N(\mathcal{E}_1 - \lambda; H_0 - V) \geqslant c(\alpha,\beta_1,L)(1+\varepsilon)^{-6/\alpha},\ \varepsilon \in (0,1). \tag{5.32}$$

The upper bound is obtained in a similar manner taking $s = (1-\varepsilon)^2$ in (5.31),

$$\limsup_{\lambda\downarrow 0} \lambda^{-\frac{1}{2}+\frac{1}{\alpha}} N(\mathcal{E}_1 - \lambda; H_0 - V) \leqslant c(\alpha,\beta_1,L)(1-\varepsilon)^{-4/\alpha},\ \varepsilon \in (0,1). \tag{5.33}$$

Letting $\varepsilon \downarrow 0$ in (5.32)-(5.33), we obtain (5.1). This completes the proof of Theorem 5.1.

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