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DETERMINING THE WAVEGUIDE CONDUCTIVITY IN A HYPERBOLIC EQUATION FROM A SINGLE MEASUREMENT ON THE LATERAL BOUNDARY

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Abstract. We consider the multidimensional inverse problem of determining the conductivity coefficient of a hyperbolic equation in an infinite cylindrical domain, from a single boundary observation of the solution. We prove Hölder stability with the aid of a Carleman estimate specifically designed for hyperbolic waveguides.

1. Statement of the problem and results.

1.1. **Introduction.** The present paper deals with the inverse problem of determining the time-independent isotropic conductivity coefficient $c : \Omega \to \mathbb{R}$ appearing in the hyperbolic partial differential equation $(\partial_t^2 - \nabla \cdot c(x)\nabla)u(x,t) = 0$, where $\Omega := \omega \times \mathbb{R}$ is an infinite cylindrical domain whose cross section ω is a bounded open subset of $\mathbb{R}^{n-1}$, $n \geq 2$. Namely, $\ell > 0$ being arbitrarily fixed, we seek Hölder stability in the identification of c in $\Omega_\ell := \omega \times (-\ell, \ell)$ from the observation of u on the lateral boundary $\Gamma_L := \partial\omega \times (-L, L)$ over the course of time $(0,T)$, for $L > \ell$ and $T > 0$ sufficiently large.

Several stability results in the inverse problem of determining one or several unknown coefficients of a hyperbolic equation from a finite number of measurements of the solution in a bounded domain are available in the mathematics literature [1, 2, 5, 6, 7, 11, 12, 18, 21]. Their derivation relies on Bukhgeim-Klibanov's method [8], which is by means of a Carleman inequality specifically designed for hyperbolic systems. More precisely, [11, 21] study the determination of the zero-th order term $p : \Omega \to \mathbb{R}$ appearing in $\partial_t^2 - \Delta + p = 0$, while [1, 7] deal with the identification of

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the speed $c : \Omega \to \mathbb{R}$ in the hyperbolic equation $\partial_t^2 - cA = 0$ where $A = A(x, D_x)$ is a second order differential operator. The case of a principal matrix term in the divergence form, arising from anisotropic media, was treated by Bellassoued, Jellali and Yamamoto in [5], using the full data (i.e. the measurements are performed on the whole boundary). Using the FBI transform Bellassoued and Yamamoto claimed logarithmic stability in [6] from arbitrarily small boundary observations. Imanuvilov and Yamamoto derived Hölder stability results in [12] by means of H^{-1} Carleman inequality, from data observation on subdomains fulfilling specific geometric assumptions. In [18] Klibanov and Yamamoto employed a different approach inspired by [17] and proved Lipschitz stability with the help of L^2 Carleman inequalities.

Similarly, numerous authors have used the Dirichlet-to-Neumann operator to claim stability in the determination of unknown coefficients of a hyperbolic equation. We refer to [4, 13, 20] for a non-exhaustive list of such references.

In all the above mentioned papers, the domain was bounded. Recently, the Bukhgeim-Klibanov method was adapted to the framework of infinite quantum cylindrical domains in [9, 15, 16]. Nevertheless, in all of these three articles the observation is taken on the infinitely extended lateral boundary of the waveguide. The approach developed in this paper is completely different in the sense that we aim to retrieve the non-compactly supported conductivity c on any arbitrary bounded subpart Ω_ℓ from one data (measured over the time span $(0, T)$) taken on a compact subset of the lateral boundary. This is made possible upon designing a suitable Carleman estimate for hyperbolic systems in finitely extended cylindrical domains Ω_L, $L > 0$. The key idea here is to distinguish between the transverse and longitudinal variables by imposing super exponential decay to the corresponding weight function with respect to the longitudinal direction, while it is expressed in the classical way with respect to the transversal variables. To our knowledge, this strategy and the corresponding stability estimate derived in this paper, are not available in the mathematical literature of inverse non-compactly supported coefficient problems.

The paper is organized as follows. Section 2 is devoted to the analysis of the direct problem associated with the hyperbolic system under study. In Section 3 we prove a global Carleman estimate specifically designed for hyperbolic systems in the cylindrical domain Ω. Finally Section 4 contains the analysis of the inverse problem and the proof of the main result.

1.2. **Settings.**

1.2.1. *Notations.* Throughout this text we write $x = (x', x_n) \in \Omega$ for every $x' := (x_1, \ldots, x_{n-1}) \in \omega$ and $x_n \in \mathbb{R}$. Further, we denote by $|y| := \left(\sum_{i=1}^m y_j^2\right)^{1/2}$ the Euclidian norm of $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, $m \in \mathbb{N}^*$, and we write

$$\mathbb{S}^{n-1} := \left\{ x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ |x'| = 1 \right\}.$$

For the sake of shortness we write ∂_j for $\partial/\partial x_j$, $j \in \mathbb{N}_{n+1}^* := \{m \in \mathbb{N}^*,\ m \leq n+1\}$. For convenience the time variable t is sometimes denoted by x_{n+1} so that $\partial_t = \partial/\partial t = \partial_{n+1}$. We set $\nabla := (\partial_1, \ldots, \partial_n)^T$, $\nabla_{x'} := (\partial_1, \ldots, \partial_{n-1})^T$ and $\nabla_{x,t} = (\partial_1, \ldots, \partial_n, \partial_t)^T$.

For any open subset D of $\mathbb{R}^m$, $m \in \mathbb{N}^*$, we note $H^p(D)$ the p-th order Sobolev space on D for every $p \in \mathbb{N}$, where $H^0(D)$ stands for $L^2(D)$. We write $\| \cdot \|_{p,D}$ for the usual norm in $H^p(D)$ and we note $H_0^1(D)$ the closure of $C_0^\infty(D)$ in the topology of $H^1(D)$.

Finally, we set $\Gamma := \partial\omega \times (-\infty, \infty)$ and for $d > 0$ we put $\Omega_d := \omega \times (-d, d)$, $Q_d := \Omega_d \times (0, T)$, $\Gamma_d := \partial\omega \times (-d, d)$ and $\Sigma_d := \partial\omega \times (-d, d) \times (0, T)$.

1.2.2. *Statement of the problem.* We examine the following initial boundary value problem (IBVP in short)

$$\begin{cases} \partial_t^2 u - \nabla \cdot c(x)\nabla u = 0 & \text{in } Q := \Omega \times (0, T), \\ u(\cdot, 0) = \theta_0,\ \partial_t u(\cdot, 0) = \theta_1 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma := \Gamma \times (0, T), \end{cases} \tag{1.1}$$

with initial data (θ_0, θ_1), where c is the unknown bounded conductivity coefficient we aim to retrieve. Given $\ell > 0$, we examine the stability issue in the inverse problem of determining c in Ω_ℓ from the observation of the normal derivative of the solution to (1.1) on Γ_L, for some $L > \ell$. This is by means of the Bukhgeim-Klibanov method imposing that the solution u to (1.1) be sufficiently smooth and appropriately bounded.

Throughout the entire text we shall suppose that c fulfills the ellipticity condition

$$c \geq c_m \text{ in } \Omega, \tag{1.2}$$

for some positive constant c_m. Notice that we may assume, and this will be systematically the case in the sequel, without restricting the generality of the foregoing, that $c_m \in (0, 1)$.

Let us now say a few words on the solution to (1.1). In order to exhibit sufficient conditions on the initial data (θ_0, θ_1) (together with the cross section ω and the conductivity c) ensuring that the solution to (1.1) is within an appropriate functional class we shall make precise further, we need to introduce the self-adjoint operator $A = A_c := -\nabla \cdot c(x)\nabla$, associated with c, generated in $L^2(\Omega)$ by the closed sesquilinear form

$$q_A[u] := \|c^{1/2}\nabla u\|_{0,\Omega}^2 = \int_\Omega c(x)|\nabla u(x)|^2 dx,\ u \in D(q_A) := H_0^1(\Omega).$$

Evidently, the operator A acts on its domain $D(A) := H_0^1(\Omega) \cap H^2(\Omega)$ as $-\nabla \cdot c(x)\nabla$. Since A is positive in $L^2(\Omega)$, by (1.2), the operator $A^{1/2}$ is well defined from the spectral theorem, and

$$D(A^{1/2}) = D(q_A) = H_0^1(\Omega).$$

For the sake of definiteness, we set $A^0 := I$ and $D(A^0) := L^2(\Omega)$, where I denotes the identity operator in $L^2(\Omega)$, and for each $m \in \mathbb{N}^*$ we put

$$A^{m/2} v := A^{(m-1)/2}(A^{1/2} v),$$

for $v \in D(A^{m/2}) := \{v \in D(A^{(m-1)/2}),\ A^{1/2} v \in D(A^{(m-1)/2})\}$.

It turns out that the linear space $D(A^{m/2})$ endowed with the scalar product

$$\langle v, w \rangle_{D(A^{m/2})} := \sum_{j=0}^{m} \langle A^{j/2} v, A^{j/2} w \rangle_{0,\Omega},$$

is Hilbertian, and it is established in Proposition 2.3 that

$$D(A^m) = \{v \in H^{2m}(\Omega);\ v, Av, \ldots, A^{p-1} v \in H_0^1(\Omega)\},\ m \in \{p - 1/2, p\},\ p \in \mathbb{N}^*, \tag{1.3}$$

provided $\partial\omega$ is C^{2m} and $c \in W^{2m-1,\infty}(\Omega)$. As a matter of fact we know from Corollary 2.4 for any natural number m, that the system (1.1) admits a unique solution

$$u \in \bigcap_{k=0}^{m+1} C^k([0,T]; H^{m+1-k}(\Omega)), \tag{1.4}$$

provided the boundary $\partial\omega$ is C^{m+1}, the conductivity $c \in W^{m,\infty}(\Omega;\mathbb{R})$ fulfills (1.2) and $(\theta_0,\theta_1) \in D(A^{(m+1)/2}) \times D(A^{m/2})$. Moreover, if $\|c\|_{W^{m,\infty}(\Omega)} \leq c_M$ for some constant $c_M > 0$, then the solution u to (1.1) satisfies the estimate

$$\sum_{k=0}^{m+1} \|u\|_{C^k([0,T];H^{m+1-k}(\Omega))} \leq C\left(\|\theta_0\|_{m+1,\Omega} + \|\theta_1\|_{m,\Omega}\right), \tag{1.5}$$

where $C > 0$ depends only on T, ω and c_M.

1.2.3. *Admissible conductivity coefficients and initial data.* In order to solve the inverse problem associated with (1.1) we seek solutions belonging to the space $\cap_{k=3}^{4} C^k([0,T]; H^{5-k}(\Omega))$. Hence we chose $m = 4$ in (1.4) and impose that c be in $W^{4,\infty}(\Omega;\mathbb{R})$ and satisfy (1.2). In what follows we note c_M a positive constant fulfilling

$$\|c\|_{W^{4,\infty}(\Omega)} \leq c_M. \tag{1.6}$$

Since our strategy is based on a Carleman estimate for the hyperbolic system (1.1), it is also required that the purely technical condition

$$a' \cdot \nabla_{x'} c \geq \mathfrak{a}_0 \text{ in } \Omega, \tag{1.7}$$

hold for some $a' = (a_1, \ldots, a_{n-1}) \in \mathbb{S}^{n-1}$ and $\mathfrak{a}_0 > 0$. Hence, given $\omega_{\#}$ an open subset in $\mathbb{R}^{n-1}$ such that $\partial\omega \subset \omega_{\#}$, we put $\mathcal{O}_* = \omega_{\#} \times \mathbb{R}$, and for $c_* \in W^{4,\infty}(\mathcal{O}_* \cap \Omega;\mathbb{R})$ satisfying

$$c_* \geq c_m \text{ and } a' \cdot \nabla_{x'} c_* \geq \mathfrak{a}_0 \text{ in } \mathcal{O}_* \cap \Omega, \tag{1.8}$$

we introduce the set $\Lambda_{\mathcal{O}_*} = \Lambda_{\mathcal{O}_*}(a', \mathfrak{a}_o, c_*, c_m, c_M)$ of admissible conductivity coefficients as

$$\Lambda_{\mathcal{O}_*} := \{c \in W^{4,\infty}(\Omega;\mathbb{R}) \text{ obeying (1.2) and (1.6)} - \text{(1.7)};\ c = c_* \text{ in } \mathcal{O}_* \cap \Omega\}. \tag{1.9}$$

Notice that the above choice of $m = 4$ dictates that (θ_0,θ_1) be taken in $D(A^{5/2}) \times D(A^2)$, i.e. that $\theta_0 \in H_0^1(\Omega) \cap H^5(\Omega)$ be such that $A\theta_0 \in H_0^1(\Omega)$ and $A^2\theta_0 \in H_0^1(\Omega)$, and that $\theta_1 \in H_0^1(\Omega) \cap H^4(\Omega)$ satisfy $A\theta_1 \in H_0^1(\Omega)$, according to (1.3).
Furthermore, it is required by the analysis of the inverse problem carried out in this article that θ_0 be in $W^{3,\infty}(\Omega)$ and satisfy

$$-(a' \cdot \nabla_{x'}\theta_0)(x) \geq \eta_0 e^{-(1+x_n^2)},\ x = (x', x_n) \in \omega_* \times \mathbb{R}, \tag{1.10}$$

for some $\eta_0 > 0$ and some open subset ω_* in $\mathbb{R}^{n-1}$, with C^2 boundary, satisfying

$$\overline{\omega \setminus (\omega_{\#} \cap \omega)} \subset \omega_* \text{ and } \overline{\omega_*} \subset \omega. \tag{1.11}$$

Notice that condition (1.10) is imposed on the open subset ω_* of the cross section and not on ω itself, since it is required that θ_0 fulfill homogeneous Dirichlet boundary conditions on $\Gamma = \partial\omega \times \mathbb{R}$.

Next, for $M_0 > 0$ such that

$$\|\theta_0\|_{W^{3,\infty}(\Omega)} + \sum_{j=0,1} \|\theta_j\|_{5-j,\Omega} \leq M_0, \tag{1.12}$$

we define the set $\Theta_{\omega_*} = \Theta_{\omega_*}(a', M_0, \eta_0)$ of admissible initial conditions (θ_0,θ_1) as

$$\Theta_{\omega_*} := \left\{(\theta_0,\theta_1) \in \left(D(A^{5/2}) \cap W^{3,\infty}(\Omega)\right) \times D(A^2),\ \text{fulfilling (1.10) and (1.12)}\right\}. \tag{1.13}$$

Having introduced all these notations we may now state the main result of this paper.

1.2.4. *Main result.* The following result claims Hölder stability in the inverse problem of determining c in Ω_ℓ, where $\ell > 0$ is arbitrary, from the knowledge of one boundary measurement of the solution to (1.1), performed on Σ_L for $L > \ell$ sufficiently large. The corresponding observation is viewed as a vector of the Hilbert space

$$\mathscr{H}(\Sigma_L) := H^3(0,T;L^2(\Gamma_L)),$$

endowed with the norm $\|v\|^2_{\mathscr{H}(\Sigma_L)} := \|v\|^2_{H^3(0,T;L^2(\Gamma_L))}$, $v \in \mathscr{H}(\Sigma_L)$.

Theorem 1.1. *Assume that $\partial\omega$ is C^5 and let $\mathcal{O}_*$ be a neighborhood of Γ in $\mathbb{R}^{n-1}$. For $a' = (a_1, \ldots, a_{n-1}) \in \mathbb{S}^{n-1}$, $\mathfrak{a}_0 > 0$, $c_m \in (0,1)$, $c_M > c_m$ and $c_* \in W^{4,\infty}(\mathcal{O}_* \cap \Omega; \mathbb{R})$ fulfilling (1.8), pick c_j, $j = 1,2$, in $\Lambda_{\mathcal{O}_*}(a', \mathfrak{a}_\mathfrak{o}, c_*, c_m, c_M)$, defined by (1.9). Further, given $M_0 > 0$, $\eta_0 > 0$, and an open subset $\omega_* \subset \mathbb{R}^{n-1}$ obeying (1.11), let $(\theta_0, 0)$ be a set of initial data in $\Theta_{\omega_*}(a', M_0, \eta_0)$, defined in (1.13).*

Then, for any $\ell > 0$, there exists $L > \ell$ and $T > 0$, such that the solution $u_j \in \bigcap_{k=0}^5 C^k([0,T], H^{5-k}(\Omega))$, $j = 1,2$, to (1.1) associated with $(\theta_0, 0)$, where c_j is substituted for c, satisfies

$$\|c_1 - c_2\|_{H^1(\Omega_\ell)} \leq C \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|^\kappa_{\mathscr{H}(\Sigma_L)}.$$

Here $\nu(x)$ is the unit outward normal vector to Γ computed at x, and $C > 0$ and $\kappa \in (0,1)$ are two constants depending only on ω, ℓ, M_0, η_0, a', $\mathfrak{a}_0$, $c_\star$, c_m and c_M.

We stress out that the measurement of the observation data is performed on Γ_L and not on the whole boundary $\partial\Omega_L$.

1.2.5. *Comments.* It is worth mentioning that there exists actual initial conditions θ_0 satisfying the conditions of Theorem 1.1. As a matter of fact, for any function $f \in H^5(\omega)$ fulfilling the condition

$$-a' \cdot \nabla_{x'} f \geq \eta_0 \text{ in } \omega_*,$$

it is apparent that $\theta_0(x) := f(x')(1 + x_n^2)^{-1}$ is lying in Θ_{ω_*}.

Moreover, we notice that the condition (1.10) imposed on θ_0 is weakening the classical non-degeneracy condition $-a' \cdot \nabla_{x'} \theta_0 \geq \eta_0$ that is usually associated with a bounded domain. The occurrence in the right hand side of (1.10) of the (super-exponentially) decreasing multiplicative term $e^{-(1+x_n^2)}$ with respect to the infinite variable x_n, is justified by the fact that there is no such thing as a square integrable function fulfilling the above mentioned classical condition in $\omega \times \mathbb{R}$. Further, we point out that (1.10) is reminiscent of the condition [16, Eq. (1.9)] imposed on the initial state by the Bukhgeim-Klibanov analysis of the inverse problem of determining the electric potential of the Schrödinger equation in an infinite cylindrical domain.

The choice $\theta_1 = 0$ in Theorem 1.1 is required by the technique used to derive the Hölder stability estimate. Indeed, the method is by means of a global hyperbolic Carleman estimate in $H^2(\Omega \times (-T,T))$, imposing that each function u_j, $j = 1,2$, extended to $\Omega \times (-T,T)$ by setting $u_j(x,t) := u_j(x,-t)$ for all $(x,t) \in \Omega \times (-T,0)$, be continuously differentiable at $t = 0$.

Notice that for simplicity, we impose that the time variable t and the longitudinal variable x_n play symmetric roles in the explicit expression (3.3) of the weight function appearing in the above mentioned hyperbolic Carleman estimate. Therefore, we shall always assume that $L = T$ in the remaining part of this text. Nevertheless, this does not restrict the generality of the result, as the general case $L \neq T$ reduces to this special one, with a slight modification of the weight function.

2. **Analysis of the direct problem.** In this section we establish existence and uniqueness results as well as regularity properties, for the solution to hyperbolic (1.1)-like IBVP systems.
We emphasize the fact these regularity results obtained for hyperbolic equations in unbounded cylindrical domains cannot be derived directly from the corresponding results existing for hyperbolic equations in bounded domains (see e. g. [10, Sect. 7.2, Theorem 6]).

2.1. **Existence and uniqueness result.** With reference to (1.1) we consider the boundary value problem

$$\left\{ \begin{array}{ll} \partial_t^2 v + Av = f & \text{in } Q \\ v(\cdot,0) = g,\ \partial_t v(\cdot,0) = h & \text{in } \Omega, \end{array} \right. \tag{2.1}$$

where f, g and h are suitable data, and we state the following existence, uniqueness and regularity result, whose proof is postponed to Appendix A and B.

Theorem 2.1. *Let m be a nonnegative integer. We assume that $g \in D(A^{(m+1)/2})$, $h \in D(A^{m/2})$, and*

$$\partial_t^k f \in C^0([0,T]; D(A^{(m-k)/2})),\ k = 0, \ldots, m.$$

Then there exists a unique solution v to (2.1), such that

$$\partial_t^k v \in C^0([0,T]; D(A^{(m+1-k)/2})),\ k = 0, 1, \ldots, m+1. \tag{2.2}$$

Moreover we have the estimate

$$\begin{aligned} &\sum_{k=0}^{m+1} \sup_{t\in[0,T]} \|\partial_t^k v(\cdot,t)\|_{D(A^{(m+1-k)/2})} \\ \leq\ & C \left(\sum_{k=0}^{m} \|\partial_t^k f\|_{C^0([0,T];D(A^{(m-k)/2}))} + \|g\|_{D(A^{(m+1)/2})} + \|h\|_{D(A^{m/2})} \right). \end{aligned} \tag{2.3}$$

Remark 2.2. The result of Theorem 2.1 is similar to the one of [10, Sect. 7.2, Theorem 6], which holds for a bounded domain. This can be seen from the characterization of the $D(A^{k/2})$ for $k = 0, \ldots, m$, displayed in Subsection 2.2. Namely, it is worth noticing that the m^{th}-order compatibility conditions [10, Sect. 7.2, Eq. (62)] imposed on f, g and h, are actually hidden in the operatorial formulation of Theorem 2.1. This will be made explicit below.

2.2. **Characterizing the domain of $A^{m/2}$ for $m \in \mathbb{N}^*$.**

Proposition 2.3. *Let $m \in \mathbb{N}^*$ and let k be either $m - 1/2$ or m. Assume that $\partial\omega$ is C^{2k} and that $c \in W^{2k-1,\infty}(\Omega)$ fullfills (1.2). Then we have*

$$D(A^k) = \{u \in H^{2k}(\Omega),\ u, Au,\ \ldots, A^{m-1}u \in H_0^1(\Omega)\}.$$

Moreover, the norm associated with $D(A^k)$ is equivalent to the usual one in $H^{2k}(\Omega)$: There exists $C(k) > 1$, depending only on k, ω, the constant c_m defined in (1.2) and $\|c\|_{W^{2k-1,\infty}(\Omega)}$, such that we have

$$C(k)^{-1}\|u\|_{D(A^k)} \leq \|u\|_{2k,\Omega} \leq C(k)\|u\|_{D(A^k)},\ u \in D(A^k).$$

In view of Theorem 2.1 and Proposition 2.3 we obtain the following result.

Corollary 2.4. *Let m be a natural number. Assume that $\partial\omega$ is C^{m+1}, that $c \in W^{m,\infty}(\Omega)$ fulfills (1.2) and that $(\theta_0, \theta_1) \in D(A^{(m+1)/2}) \times D(A^{m/2})$. Then the initial boundary value problem (1.1) admits a unique solution*

$$u \in \bigcap_{k=0}^{m+1} C^k([0,T]; H^{m+1-k}(\Omega)).$$

Moreover, we have

$$\sum_{k=0}^{m+1} \|u\|_{C^k([0,T];H^{m+1-k}(\Omega))} \leq C\left(\|\theta_0\|_{m+1,\Omega} + \|\theta_1\|_{m,\Omega}\right), \tag{2.4}$$

for some constant $C > 0$ depending only on T, ω and $\|c\|_{W^{m,\infty}(\Omega)}$.

The main benefit of using the successive powers of the operator $A^{1/2}$ in the formulation of Theorem 2.1 and Corollary 2.4 lies in its simplicity. Nevertheless, to make these statements more explicit, we introduce for any $u \in H_0^1(\Omega)$ the sequence $(u_n)_{n\in\mathbb{N}}$ by setting

$$u_0 := u \text{ and } u_n := -\nabla \cdot c\nabla u_{n-1} \text{ for } n \in \mathbb{N}^*,$$

and, given $m \in \mathbb{N}^*$, we say that u satisfies the m^{th}-order compatibility condition (with respect to c) if

$$u_n \in H_0^1(\Omega) \text{ for } n = 0, 1, \ldots, m-1.$$

Therefore, for every $m \in \mathbb{N}^*$, we see from Proposition 2.3 that

$$D(A^m) = \{u \in H^{2m}(\Omega),\ u \text{ satisfies the } m^{\text{th}}\text{-order compatibility condition}\},$$

and

$$D(A^{m-1/2}) = \{u \in H^{2m-1}(\Omega),\ u \text{ satisfies the } m^{\text{th}}\text{-order compatibility condition}\}.$$

3. **Global Carleman estimate for hyperbolic equations in cylindrical domains.** In this section we establish a global Carleman estimate for the system (1.1). To this purpose we start by time-symmetrizing the solution u of (1.1). Namely, we put

$$u(x,t) := u(x,-t),\ x \in \Omega,\ t \in (-T,0). \tag{3.1}$$

Under the conditions of Theorem 1.1, and since $\theta_1 = 0$, it is not hard to check that

$$u \in \bigcap_{k=3}^{4} C^k([-T,T]; H^{5-k}(\Omega)).$$

With a slight abuse of notations we put $Q := \Omega \times (-T,T)$, $\Sigma := \Gamma \times (-T,T)$ and $Q_L := \Omega_L \times (-T,T)$, $\Sigma_L := \partial\omega \times (-L,L) \times (-T,T)$ for any $L > 0$, in the remaining part of this text.

3.1. **The case of second order hyperbolic operators.** In view of establishing a Carleman estimate for the operator

$$\mathscr{H} = \mathscr{H}(x,t,\partial) := \partial_t^2 - \nabla \cdot c(x) \nabla + R, \tag{3.2}$$

where R is a first-order partial differential operator with $L^\infty(Q)$ coefficients, we define for $a' \in \mathbb{S}^{n-1}$ fulfilling condition (1.7), the following weight functions:

$$\psi(x,t) = \psi_\delta(x,t) := |x'-\delta a'|^2 - x_n^2 - t^2 \text{ and } \varphi(x,t) = \varphi_{\delta,\gamma}(x,t) := \mathrm{e}^{\gamma \psi(x,t)},\ (x,t) \in Q, \tag{3.3}$$

for all $\delta > 0$ and $\gamma > 0$.

Notice that the cylindrical geometry of Ω is reflected in the expression of these weight functions through the fact that the longitudinal variable x_n plays a role different from that of the transverse variable x' within (3.3). We stress out that this feature, which is a cornerstone of the analysis of the inverse coefficient problem carried out in this text, is specific to waveguides, as the classical weight function ψ (see e.g. [14, Eq. (3.4.1)]) used by hyperbolic Carleman estimates in a domain of general shape in $\mathbb{R}^n$, is quite different from the one given by (3.3).

We turn now to establishing the following Carleman estimate for the operator $\mathscr{H}$.

Proposition 3.1. *Let $\mathscr{H}$ be defined by (3.2), where c obeys (1.2) and (1.6)-(1.7), and let ℓ be positive. Then there exist $\delta_0 > 0$ and $\gamma_0 > 0$, such that for all $\delta \geq \delta_0$ and $\gamma \geq \gamma_0$, there exist $L > \ell$, $T > 0$, and $s_0 > 0$, for which the estimate*

$$s \sum_{j=0,1} s^{2(1-j)} \| \mathrm{e}^{s\varphi} \nabla_{x,t}^j v \|_{0,Q_L}^2 \leq C \left(\| \mathrm{e}^{s\varphi} \mathscr{H} v \|_{0,Q_L}^2 + s \sum_{j=0,1} s^{2(1-j)} \| \mathrm{e}^{s\varphi} \nabla_{x,t}^j v \|_{0,\partial Q_L}^2 \right), \tag{3.4}$$

holds for any $s \geq s_0$ and $v \in H^2(Q_L)$. Here C is a positive constant depending only on ω, a', $\mathfrak{a}_0$, δ_0, γ_0, s_0, c_m and c_M.

Moreover there exists a constant $d_\ell > 0$, depending only on ω, ℓ, δ_0 and γ_0, such that the weight function φ defined by (3.3) satisfies

$$\varphi(x', x_n, 0) \geq d_\ell,\ (x', x_n) \in \overline{\omega} \times [-\ell, \ell], \tag{3.5}$$

and there exist $\epsilon \in (0, (L-\ell)/2)$ and $\nu_0 > 0$ so small that we have:

$$\max_{x \in \overline{\omega} \times [-L,L]} \varphi(x', x_n, t) \leq \tilde{d}_\ell := d_\ell \mathrm{e}^{-\gamma \nu_0^2},\ |t| \in [T-2\epsilon, T], \tag{3.6}$$

$$\max_{(x',t) \in \overline{\omega} \times [-T,T]} \varphi(x', x_n, t) \leq \tilde{d}_\ell,\ |x_n| \in [L-2\epsilon, L]. \tag{3.7}$$

Proof. The proof is divided in three parts. The derivation of the Carleman estimate (3.4) essentially boils down to [14, Theorem 3.2.1']. It consists in proving pseudo-convexity of ψ for the second order operator $\mathscr{H}$, i.e. that, first, the principal part $\mathscr{H}_2$ of $\mathscr{H}$ fulfills (3.12), and, second, that under the conditions (3.14)-(3.15), the function J defined by (3.13), is positive in $\overline{Q}_L \times \left(\mathbb{R}^{n+1} \setminus \{0\}\right)$. This is achieved in the *second part* of the proof by means of two intermediate estimates, (3.10) and (3.11). Actually, much of the technical work of this proof is carried out in the *first part*, where sufficient conditions on the parameters L and T, ensuring (3.10)-(3.11), are exhibited. Finally, we establish the estimates (3.5)–(3.7) in the *third part* of the proof.

First part: Definition of δ_0, L and T. Bearing in mind that

$$|x' - \delta a'|^2 - |y' - \delta a'|^2 = |x'| - |y'|^2 - 2\delta a' \cdot (x' - y'),\ x', y' \in \omega,$$

we see that $\sup_{x'\in\omega}|x'-\delta a'|^2-\inf_{x'\in\omega}|x'-\delta a'|^2\leq|\omega|(|\omega|+4\delta|a'|)$, for every $\delta>0$, where $|\omega|:=\sup_{x'\in\overline{\omega}}|x'|$. Hence the function

$$g_\ell(\delta)=\left(\sup_{x'\in\omega}|x'-\delta a'|^2-\inf_{x'\in\omega}|x'-\delta a'|^2+\ell^2\right)^{1/2} \tag{3.8}$$

scales at most like $\delta^{1/2}$, proving that there exists $\delta_0>0$ so large that

$$\delta\mathfrak{a}_0>\left(\left(1+\frac{2\sqrt{n}}{{c_m}^{1/2}}\right)g_\ell(\delta)+\sqrt{n-1}|\omega|+2\right)c_M+2,\ \delta\geq\delta_0. \tag{3.9}$$

This technical condition links geometric parameters and conditions on the conductivity c and will be useful via (3.10) in the second part of the proof of the proposition 3.1. Further, since ω is bounded and $a'\neq 0_{\mathbb{R}^{n-1}}$ by (1.7), we may as well assume upon possibly enlarging δ_0, that we have in addition $c_m^{1/2}\inf_{x'\in\overline{\omega}}|x'-\delta a'|>g_\ell(\delta)$ for all $\delta\geq\delta_0$. This and (3.9) yield that there exists $\vartheta>0$ so small that the two following inequalities

$$\delta\mathfrak{a}_0-\left(L+\sqrt{n-1}|\omega|+2\left(1+\frac{\sqrt{n}T}{{c_m}^{1/2}}\right)\right)c_M-2>0, \tag{3.10}$$

and

$$c_m^{1/2}\inf_{x'\in\overline{\omega}}|x'-\delta a'|>T, \tag{3.11}$$

hold simultaneously for every L and T in $(g_\ell(\delta),g_\ell(\delta)+\vartheta)$, uniformly in $\delta\geq\delta_0$.
Second part: Proof of (3.4). We first introduce the following notations, we shall use in the remaining part of the proof. For notational simplicity we put $\mathbf{x}:=(x,t)$ for $(x,t)\in Q_L$ and $\nabla_{\mathbf{x}}=(\partial_1,\ldots,\partial_n,\partial_{n+1})^T$. We also write $\xi'=(\xi_1,\ldots,\xi_{n-1})^T\in\mathbb{R}^{n-1}$, $\xi=(\xi_1,\ldots,\xi_n)^T\in\mathbb{R}^n$ and $\tilde{\xi}=(\xi_1,\ldots,\xi_n,\xi_{n+1})^T\in\mathbb{R}^{n+1}$. We call $\mathscr{H}_2$ the principal part of the operator $\mathscr{H}$, that is $\mathscr{H}_2=\mathscr{H}_2(\mathbf{x},\partial):=\partial_t^2-c(x)\Delta$, and denote its symbol by $\mathscr{H}_2(x,\tilde{\xi}):=c(x)|\xi|^2-\xi_{n+1}^2$, where $|\xi|=\left(\sum_{j=1}^n\xi_j^2\right)^{1/2}$. Since $\mathscr{H}_2(x,\nabla_{\mathbf{x}}\psi(\mathbf{x}))=4\left(c(x)(|x'-\delta a'|^2+x_n^2)-x_{n+1}^2\right)$ for every $\mathbf{x}\in\overline{Q}_L$, we have

$$\mathscr{H}_2(x,\nabla_{\mathbf{x}}\psi(\mathbf{x}))>0,\ \mathbf{x}\in\overline{Q}_L, \tag{3.12}$$

by (3.11). For all $\mathbf{x}\in\overline{Q}_L$ and $\tilde{\xi}\in\mathbb{R}^{n+1}$, put

$$J(\mathbf{x},\tilde{\xi})=J=\sum_{j,k=1}^{n+1}\frac{\partial\mathscr{H}_2}{\partial\xi_j}\frac{\partial\mathscr{H}_2}{\partial\xi_k}\partial_j\partial_k\psi+\sum_{j,k=1}^{n+1}\left(\left(\partial_k\frac{\partial\mathscr{H}_2}{\partial\xi_j}\right)\frac{\partial\mathscr{H}_2}{\partial\xi_k}-(\partial_k\mathscr{H}_2)\frac{\partial^2\mathscr{H}_2}{\partial\xi_j\partial\xi_k}\right)\partial_j\psi, \tag{3.13}$$

where, for the sake of shortness, we write ∂_j, $j\in\mathbb{N}_{n+1}^*$, instead of $\partial/\partial x_j$, and x_{n+1} stands for t. Assuming that

$$\mathscr{H}_2(x,\tilde{\xi})=c(x)|\xi|^2-\xi_{n+1}^2=0,\ x\in\overline{\Omega},\ \tilde{\xi}\in\mathbb{R}^{n+1}\backslash\{0\}, \tag{3.14}$$

and that

$$\begin{aligned}\nabla_{\tilde{\xi}}\mathscr{H}_2(x,\tilde{\xi})\cdot\nabla_{\mathbf{x}}\psi(\mathbf{x}) &= 4\left[c(x)(\xi'\cdot(x'-\delta a')-\xi_n x_n)+\xi_{n+1}x_{n+1}\right]\\ &= 0,\ \mathbf{x}\in\overline{Q}_L,\ \tilde{\xi}\in\mathbb{R}^{n+1}\backslash\{0\},\end{aligned} \tag{3.15}$$

we shall prove that $J(\mathbf{x},\tilde{\xi})>0$ for any $(\mathbf{x},\tilde{\xi})\in\overline{Q}_L\times\left(\mathbb{R}^{n+1}\setminus\{0\}\right)$. To this end we notice that the first sum in the right hand side of (3.13) reads

$$\langle\mathrm{Hess}(\psi)\nabla_{\tilde{\xi}}\mathscr{H}_2,\nabla_{\tilde{\xi}}\mathscr{H}_2\rangle=8\left(c^2(|\xi'|^2-\xi_n^2)-\xi_{n+1}^2\right),$$

that the sum in $\sum_{j,k=1}^{n+1}\left(\left(\partial_k \frac{\partial \mathscr{H}_2}{\partial \xi_j}\right)\frac{\partial \mathscr{H}_2}{\partial \xi_k}-(\partial_k \mathscr{H}_2)\frac{\partial^2 \mathscr{H}_2}{\partial \xi_j \partial \xi_k}\right)\partial_j \psi$ can actually be taken over $(j,k)\in(\mathbb{N}_n^*)^2$ only, since $\left(\partial_k \frac{\partial \mathscr{H}_2}{\partial \xi_j}\right)\frac{\partial \mathscr{H}_2}{\partial \xi_k}-(\partial_k \mathscr{H}_2)\frac{\partial^2 \mathscr{H}_2}{\partial \xi_j \partial \xi_k}=0$ if either j or k is equal to $n+1$, and hence that

$$\begin{aligned}
&\sum_{j,k=1}^{n+1}\left(\left(\partial_k \frac{\partial \mathscr{H}_2}{\partial \xi_j}\right)\frac{\partial \mathscr{H}_2}{\partial \xi_k}-(\partial_k \mathscr{H}_2)\frac{\partial^2 \mathscr{H}_2}{\partial \xi_j \partial \xi_k}\right)\partial_j \psi \\
= \;& 2c\sum_{j,k=1}^{n}\left(\left(2\xi_j\xi_k-|\xi|^2\frac{\partial \xi_j}{\partial \xi_k}\right)\partial_k c\right)\partial_j\psi \\
= \;& 2c\left(2(\nabla c\cdot\xi)(\nabla\psi\cdot\xi)-(\nabla c\cdot\nabla\psi)|\xi|^2\right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
J &= 2\left[4\left(c^2(|\xi'|^2-\xi_n^2)-\xi_{n+1}^2\right)+2c(\nabla c\cdot\xi)(\nabla\psi\cdot\xi)-c(\nabla c\cdot\nabla\psi)|\xi|^2\right] \\
&= 4\left[2c^2(|\xi'|^2-\xi_n^2)-\left(2+(x'-\delta a')\cdot\nabla_{x'}c-x_n\partial_n c\right)\xi_{n+1}^2-2x_{n+1}\xi_{n+1}\nabla c\cdot\xi\right],
\end{aligned}$$

from (3.14)-(3.15). Further, in view of (3.14) we have

$$c^2(|\xi'|^2-\xi_n^2)\ge -c^2|\xi|^2\ge -c\xi_{n+1}^2 \text{ and } |\nabla c\cdot\xi|\le|\nabla c||\xi|\le(|\nabla c|/c^{1/2})|\xi_{n+1}|,$$

whence

$$J\ge 4\left[\delta a'\cdot\nabla_{x'}c-\left(x'\cdot\nabla_{x'}c-x_n\partial_n c+2c+2T\frac{|\nabla c|}{c^{1/2}}+2\right)\right]\xi_{n+1}^2. \tag{3.16}$$

Here we used the fact that $x_{n+1}=t\in[0,T]$. Due to (1.6)-(1.7), the right hand side of (3.16) is lower bounded, up to the multiplicative constant $4\xi_{n+1}^2$, by the left hand side of (3.10). Since ξ_{n+1} is non zero by (1.6) and (3.14), then we obtain $J(\mathbf{x},\tilde{\xi})>0$ for all $(\mathbf{x},\tilde{\xi})\in\overline{Q}_L\times\left(\mathbb{R}^{n+1}\setminus\{0\}\right)$. With reference to (3.12), we may apply [14, Theorem 3.2.1'], getting two constants $s_0=s_0(\gamma)>0$ and $C>0$ such that (3.4) holds for any $s\ge s_0$ and $v\in H^2(Q_L)$.

Third part: Proof of (3.5)–(3.7). First, (3.5) follows readily from (3.3), with $d_\ell:=\mathrm{e}^{\gamma\beta_\ell}$ and $\beta_\ell:=\inf_{x'\in\omega}|x'-\delta a'|^2-\ell^2$. Next, for $\nu_0\in(0,\vartheta)$ arbitrarily fixed, we put

$$L=T=g_\ell(\delta)+\nu_0. \tag{3.17}$$

Notice for further reference from (3.8), (3.11) and (3.17), that we have

$$\beta_\ell\ge\frac{g_\ell(\delta)^2}{c_m}-\ell^2\ge\left(\frac{1-c_m}{c_m}\right)\ell^2>0, \tag{3.18}$$

since $c_m\in(0,1)$, by assumption. Similarly, as

$$T^2>g_\ell(\delta)^2+\nu_0^2=\sup_{x'\in\omega}|x'-\delta a'|^2-(\beta_\ell^2-\nu_0^2),$$

we deduce from (3.3) that

$$\varphi(x',x_n,\pm T)\le\mathrm{e}^{\gamma\left(\sup_{x'\in\omega}|x'-\delta a'|^2-x_n^2-T^2\right)}<\mathrm{e}^{\gamma(\beta_\ell-\nu_0^2)}\mathrm{e}^{-\gamma x_n^2},\ (x',x_n)\in\overline{\omega}\times[-L,L].$$

With reference to (3.18) we may thus choose $\epsilon\in(0,(L-\ell)/2)$ so small that

$$\varphi(x',x_n,t)\le d_\ell\mathrm{e}^{-\gamma\nu_0^2}\mathrm{e}^{-\gamma x_n^2},\ (x',x_n)\in\overline{\omega}\times[-L,L],\ |t|\in[T-2\epsilon,T],$$

which entails (3.6). Finally, since t and x_n play symmetric roles in (3.3), and since $T=L$, we obtain (3.7) by substituting (T,t) for (L,x_n) in (3.6). $\square$

3.2. **A Carleman estimate for the system (1.1).** In this subsection we derive from Proposition 3.1 a global Carleman estimate for the solution to the boundary value problem

$$\begin{cases} \partial_t^2 u - \nabla\cdot c(x)\nabla u = f & \text{in } Q \\ u = 0 & \text{on } \Sigma, \end{cases} \tag{3.19}$$

where $f \in L^2(Q)$. To this purpose we introduce a cut-off function $\chi \in C^2(\mathbb{R};[0,1])$, such that

$$\chi(x_n) := \begin{cases} 1 & \text{if } |x_n| < L - 2\epsilon, \\ 0 & \text{if } |x_n| \geq L - \epsilon, \end{cases} \tag{3.20}$$

where ϵ is the same as in Proposition 3.1, and we set

$$u_\chi(x,t) := \chi(x_n)u(x,t) \text{ and } f_\chi(x,t) := \chi(x_n)f(x,t),\ (x,t)\in Q.$$

Corollary 3.2. *Let $f \in L^2(Q)$. Then, under the conditions of Proposition 3.1, there exist two constants $s_* > 0$ and $C > 0$, depending only on ω, ℓ, M_0, η_0, a', $\mathfrak{a}_0$, c_m and c_M, such that the estimate*

$$\begin{aligned} & s\sum_{j=0,1} s^{2(1-j)}\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j u\|_{0,Q_L}^2 \\ \leq\ & C\left(\|\mathrm{e}^{s\varphi}f\|_{0,Q_L}^2 + s^3\mathrm{e}^{2s\tilde{d}_\ell}\|u\|_{1,Q_L}^2 + s\sum_{j=0,1} s^{2(1-j)}\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j u_\chi\|_{0,\Sigma_L}^2\right), \end{aligned}$$

holds for any solution $u \in H^2(Q)$ to (3.19), uniformly in $s \geq s_*$.

Proof. Since u is solution to (3.19) we have

$$\partial_t^2 u_\chi - \nabla\cdot c(x)\nabla u_\chi = f_\chi + R_1 u \text{ in } Q,$$

where

$$R_1 = R_1(x,\partial) := [\chi, \nabla\cdot c\nabla] = -(c\Delta\chi + \nabla c\cdot\nabla\chi + 2c\nabla\chi\cdot\nabla), \tag{3.21}$$

is a first-order differential operator. Therefore, the function $v(x,t) := \eta(t)u_\chi(x,t)$, where $\eta \in C^2(\mathbb{R};[0,1])$ is such that

$$\eta(t) := \begin{cases} 1 & \text{if } |t| < T - 2\epsilon, \\ 0 & \text{if } |t| \geq T - \epsilon, \end{cases} \tag{3.22}$$

satisfies $\partial_t^2 v - \nabla\cdot c\nabla v = g := \eta f_\chi + \eta R_1 u + \eta'' u_\chi + 2\eta'\partial_t u_\chi$ in Q.

Moreover, as $v \in H^2(Q_L)$, we may apply Proposition 3.1, getting

$$s\sum_{j=0,1} s^{2(1-j)}\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j v\|_{0,Q_L}^2 \leq C\left(\|\mathrm{e}^{s\varphi}g\|_{0,Q_L}^2 + s\sum_{j=0,1} s^{2(1-j)}\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j v\|_{0,\partial Q_L}^2\right). \tag{3.23}$$

Further, bearing in mind that $\partial Q_L = \Sigma_L \cup (\omega\times\{\pm L\}\times(-T,T)) \cup (\Omega_L \times \{\pm T\})$, we deduce from the vanishing of $v(\cdot,\pm L,\cdot)$ and $\nabla_{x,t}v(\cdot,\pm L,\cdot)$ in $\omega\times(-T,T)$, and the one of $v(\cdot,\pm T)$ and $\nabla_{x,t}v(\cdot,\pm T)$ in Ω_L, that

$$\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j v\|_{0,\partial Q_L} = \|\mathrm{e}^{s\varphi}\nabla_{x,t}^j v\|_{0,\Sigma_L},\ j = 0,1. \tag{3.24}$$

Next we know from (3.7) and (3.21) that

$$\|\mathrm{e}^{s\varphi}\eta R_1 u\|_{0,Q_L} \leq C\mathrm{e}^{s\tilde{d}_\ell}\|u\|_{L^2(-T,T;H^1(\Omega_L))}, \tag{3.25}$$

and from (3.6) that

$$\|\mathrm{e}^{s\varphi}(\eta'' u_\chi + 2\eta'\partial_t u_\chi)\|_{0,Q_L} \leq C\mathrm{e}^{s\tilde{d}_\ell}\|u_\chi\|_{H^1(-T,T;L^2(\Omega_L))}. \tag{3.26}$$

Hence, putting (3.23)–(3.26) together, we find out that

$$\begin{aligned} & s \sum_{j=0,1} s^{2(1-j)} \|\mathrm{e}^{s\varphi} \nabla_{x,t}^j v\|_{0,Q_L}^2 \\ \leq \quad & C \left(\|\mathrm{e}^{s\varphi} f\|_{0,Q_L}^2 + s \sum_{j=0,1} s^{2(1-j)} \|\mathrm{e}^{s\varphi} \nabla_{x,t}^j u_\chi\|_{0,\Sigma_L}^2 + \mathrm{e}^{2s\tilde{d}_\ell} \|u\|_{1,Q_L}^2 \right) \end{aligned} \tag{3.27}$$

The next step of the proof involves noticing from (3.6) that $\|\mathrm{e}^{s\varphi}(1-\eta)\nabla_x^j u_\chi\|_{0,Q_L} \leq \mathrm{e}^{s\tilde{d}_\ell} \|\nabla_x^j u_\chi\|_{0,Q_L}$ for $j = 0, 1$, hence

$$\begin{aligned} \|\mathrm{e}^{s\varphi} \nabla_x^j u_\chi\|_{0,Q_L} & \leq \|\mathrm{e}^{s\varphi}(1-\eta)\nabla_x^j u_\chi\|_{0,Q_L} + \|\mathrm{e}^{s\varphi} \nabla_x^j v\|_{0,Q_L} \\ & \leq \mathrm{e}^{s\tilde{d}_\ell} \|\nabla_x^j u_\chi\|_{0,Q_L} + \|\mathrm{e}^{s\varphi} \nabla_x^j v\|_{0,Q_L}, \ j = 0, 1. \end{aligned} \tag{3.28}$$

Furthermore, by combining the identity $\partial_t u_\chi = (1-\eta)\partial_t u_\chi + \eta \partial_t u_\chi = (1-\eta)\partial_t u_\chi - \eta' \partial_t u_\chi + \partial_t v$ with (3.6), we get that

$$\begin{aligned} \|\mathrm{e}^{s\varphi} \partial_t u_\chi\|_{0,Q_L} & \leq \|\mathrm{e}^{s\varphi}(1-\eta)\partial_t u_\chi\|_{0,Q_L} + \|\mathrm{e}^{s\varphi} \eta' u_\chi\|_{0,Q_L} + \|\mathrm{e}^{s\varphi} \partial_t v\|_{0,Q_L} \\ & \leq \mathrm{e}^{s\tilde{d}_\ell} \left(\|\partial_t u_\chi\|_{0,Q_L} + \|\eta'\|_{L^\infty(-T,T)} \|u_\chi\|_{0,Q_L} \right) + \|\mathrm{e}^{s\varphi} \partial_t v\|_{0,Q_L}, \end{aligned}$$

which, together with (3.28), yields

$$\begin{aligned} & \sum_{j=0,1} s^{2(1-j)} \|\mathrm{e}^{s\varphi} \nabla_{x,t}^j u_\chi\|_{0,Q_L}^2 \\ \leq \quad & C \sum_{j=0,1} s^{2(1-j)} \left(\mathrm{e}^{2s\tilde{d}_\ell} \|\nabla_{x,t}^j u_\chi\|_{0,Q_L}^2 + \|\mathrm{e}^{s\varphi} \nabla_{x,t}^j v\|_{0,Q_L}^2 \right). \end{aligned} \tag{3.29}$$

Similarly, using (3.7), we derive from the identity $\partial_t^j u = \partial_t^j u_\chi + (1-\chi)\partial_t^j u$ for $j = 0, 1$, that

$$\|\mathrm{e}^{s\varphi} \partial_t^j u\|_{0,Q_L} \leq \|\mathrm{e}^{s\varphi} \partial_t^j u_\chi\|_{0,Q_L} + \mathrm{e}^{s\tilde{d}_\ell} \|\partial_t^j u\|_{0,Q_L}, \ j = 0, 1,$$

and from $\nabla_x u = \nabla_x u_\chi + (1-\chi)\nabla_x u - \chi'(0, \ldots, 0, u)^T$, that

$$\|\mathrm{e}^{s\varphi} \nabla_x u\|_{0,Q_L} \leq \|\mathrm{e}^{s\varphi} \nabla_x u_\chi\|_{0,Q_L} + \mathrm{e}^{s\tilde{d}_\ell} \left(\|\nabla_x u\|_{0,Q_L} + \|\chi'\|_{L^\infty(-L,L)} \|u\|_{0,Q_L} \right).$$

As a consequence we have

$$\sum_{j=0,1} s^{2(1-j)} \|\mathrm{e}^{s\varphi} \nabla_{x,t}^j u\|_{0,Q_L}^2 \leq C \sum_{j=0,1} s^{2(1-j)} \left(\mathrm{e}^{2s\tilde{d}_\ell} \|\nabla_{x,t}^j u\|_{0,Q_L}^2 + \|\mathrm{e}^{s\varphi} \nabla_{x,t}^j u_\chi\|_{0,Q_L}^2 \right). \tag{3.30}$$

Finally we obtain the desired result by gathering (3.27) and (3.29)-(3.30). □

4. **Inverse problem.** In this section we prove the statement of Theorem 1.1.

4.1. **Linearized inverse problem and preliminary estimate.** In this subsection we introduce the linearized inverse problem associated with (1.1) and relate the first Sobolev norm of the conductivity to some suitable initial condition of this boundary problem.

Namely, given $c_i \in \Lambda_\Gamma$ for $i = 1, 2$, we note u_i the solution to (1.1) where c_i is substituted for c, suitably extended to $(-T, 0)$ in accordance with (3.1). Thus, putting

$$c := c_1 - c_2 \text{ and } f_c := \nabla \cdot (c \nabla u_2), \tag{4.1}$$

it is clear from (1.1) that the function $u := u_1 - u_2$ is solution to the linearized system

$$\begin{cases} \partial_t^2 u - \nabla \cdot (c_1 \nabla u) = f_c & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{4.2}$$

By differentiating k-times (4.2) with respect to t, for $k \in \mathbb{N}^*$ fixed, we see that $u^{(k)} := \partial_t^k u$ is solution to

$$\begin{cases} \partial_t^2 u^{(k)} - \nabla \cdot (c_1 \nabla u^{(k)}) = f_c^{(k)} & \text{in } Q \\ u^{(k)} = 0 & \text{on } \Sigma, \end{cases} \tag{4.3}$$

with $f_c^{(k)} := \partial_t^k f_c = \nabla \cdot (c \nabla u_2^{(k)})$, where $u_2^{(k)}$ stands for $\partial_t^k u_2$.

We stick with the notations of Corollary 3.2. In particular, for any function v, we denote χv by v_χ, where χ is defined in (3.20). Upon multiplying both sides of the identity (4.3) by χ, we obtain that

$$\begin{cases} \partial_t^2 u_\chi^{(k)} - \nabla \cdot (c_1 \nabla u_\chi^{(k)}) = f_{c_\chi}^{(k)} - g_k & \text{in } Q \\ u_\chi^{(k)} = 0 & \text{on } \Sigma, \end{cases} \tag{4.4}$$

with

$$f_{c_\chi} := \nabla \cdot (c_\chi \nabla u_2) \text{ and } g_k := \nabla \cdot (c_1 (\nabla \chi) u^{(k)}) + c_1 \nabla \chi \cdot \nabla u^{(k)} + c \nabla \chi \cdot \nabla u_2^{(k)}. \tag{4.5}$$

Notice that g_k is supported in $\tilde{\Omega}_\epsilon := \{x = (x', x_n),\ x' \in \omega \text{ and } |x_n| \in (L - 2\epsilon, L - \epsilon)\}$.

Having said that we may now upper bound, up to suitable additive and multiplicative constants, the $e^{s\varphi(\cdot,0)}$-weighted first Sobolev norm of the conductivity c_χ in Ω_L, by the corresponding norm of the initial condition $u_\chi^{(2)}(\cdot, 0)$.

Lemma 4.1. *Let u be the solution to the linearized problem (4.2) and let χ be defined by (3.20). Then there exist two constants $s_* > 0$ and $C > 0$, depending only on ω, ε and the constant M_0 defined by (1.12), such that the estimate*

$$\sum_{j=0,1} \| e^{s\varphi(\cdot,0)} \nabla^j c_\chi \|_{0,\Omega_L}^2 \leq C s^{-1} \left(\sum_{j=0,1} \| e^{s\varphi(\cdot,0)} \nabla^j u_\chi^{(2)}(\cdot, 0) \|_{0,\Omega_L}^2 + e^{2s\tilde{d}_\ell} \right),$$

holds for all $s \geq s_*$.

Proof. Let Ω_* be an open subset of $\mathbb{R}^n$ with C^2 boundary, such that

$$\omega_* \times (-(L - \epsilon), L - \epsilon) \subset \Omega_* \subset \omega_* \times (-L, L), \tag{4.6}$$

where ϵ is defined by Proposition 3.1. We notice from (1.9) and (3.20) that $\partial_i^j c_\chi \in H_0^1(\Omega_*)$ for all $i \in \mathbb{N}_n^*$ and $j = 0, 1$.

Further, with reference to (1.10) we may assume upon possibly enlarging $\delta \in [\delta_0, +\infty)$, where δ_0 is the same as in Proposition 3.1, that we have

$$|\nabla \theta_0 \cdot (x_1 - \delta a_1, \dots, x_{n-1} - \delta a_{n-1}, -x_n)| \geq \mu_0 > 0,\ x \in \Omega_*,$$

with some constant $\mu_0 > 0$. Thus applying [12, Proposition 2.2] [1] with $D = \Omega_*$, $P(x, \partial) v = \nabla \cdot ((\nabla \theta_0) v)$ and $v = \partial_i^j c_\chi \in H_0^1(\Omega_*)$ since $\chi(x_n) = 0$ for $x_n \geq L - \epsilon$,

[1] Let D be a bounded open subset of $\mathbb{R}^n$, $n \geq 1$, with C^2 boundary, and consider the first-order operator $P(x, \partial) := \sum_{i=1}^n p_i(x) \partial_i + p_0(x)$, where $p_0 \in C^0(\overline{D})$ and $p := (p_1, \dots, p_n) \in C^1(\overline{D})^n$. Assume that

$$|p(x) \cdot (x_1 - \delta a_1, \dots, x_{n-1} - \delta a_{n-1}, -x_n)| \geq \mathfrak{p}_m,\ x \in \overline{D},$$

for $i \in \mathbb{N}_n^*$ and $j=0,1$, we obtain that

$$s\|e^{s\varphi(\cdot,0)}\partial_i^j c_\chi\|_{0,\Omega_*}^2 \leq C\|e^{s\varphi(\cdot,0)}\nabla\cdot((\partial_i^j c_\chi)\nabla\theta_0)\|_{0,\Omega_*}^2,\ i\in\mathbb{N}_n^*.\ j=0,1. \qquad (4.7)$$

Since $c_\chi(x',x_n)=0$ a.e. in $\omega_*\times((-L,-(L-\epsilon))\cup(L-\epsilon,L))$ by (3.20), we have $\|e^{s\varphi(\cdot,0)}\partial_i^j c_\chi\|_{0,\Omega_*}=\|e^{s\varphi(\cdot,0)}\partial_i^j c_\chi\|_{0,\Omega_L}$ for each $i\in\mathbb{N}_n^*$ and $j=0,1$, from (4.6). We derive from this and (4.7) that

$$s\|e^{s\varphi(\cdot,0)}\partial_i^j c_\chi\|_{0,\Omega_L}^2 \leq C\|e^{s\varphi(\cdot,0)}\nabla\cdot((\partial_i^j c_\chi)\nabla\theta_0)\|_{0,\Omega_L}^2,\ i\in\mathbb{N}_n^*,\ j=0,1. \qquad (4.8)$$

Further, taking $t=0$ in the first line of (4.4) with $k=0$, we get that

$$\nabla\cdot(c_\chi\nabla\theta_0)=u_\chi^{(2)}(\cdot,0)+c\nabla\chi\cdot\nabla\theta_0. \qquad (4.9)$$

From this, (3.7) and (4.8) it then follows that

$$\begin{aligned} s\|e^{s\varphi(\cdot,0)}c_\chi\|_{0,\Omega_L}^2 &\leq C\left(\|e^{s\varphi(\cdot,0)}u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_L}^2+\|e^{s\varphi(\cdot,0)}c\nabla\chi\cdot\nabla\theta_0\|_{0,\Omega_L}^2\right)\\ &\leq C\left(\|e^{s\varphi(\cdot,0)}u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_L}^2+e^{2s\tilde{d}_\ell}\right). \qquad (4.10)\end{aligned}$$

Similarly, since $\nabla\cdot((\partial_i c_\chi)\nabla\theta_0)=\partial_i\nabla\cdot(c_\chi\theta_0)-\nabla\cdot(c_\chi\nabla\partial_i\theta_0)$ for every $i\in\mathbb{N}_n^*$, we derive from (4.9) that

$$\nabla\cdot((\partial_i c_\chi)\nabla\theta_0)=\partial_i u_\chi^{(2)}(\cdot,0)+\partial_i(c\nabla\theta_0\cdot\nabla\chi)-\nabla c_\chi\cdot\nabla\partial_i\theta_0-c_\chi\Delta\partial_i\theta_0.$$

As a consequence we have,

$$\begin{aligned} &\|e^{s\varphi(\cdot,0)}\nabla\cdot((\partial_i c_\chi)\nabla\theta_0)\|_{0,\Omega_L}^2\\ \leq\ & C\left(\|e^{s\varphi(\cdot,0)}\partial_i u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_l}^2+\sum_{j=0,1}\|e^{s\varphi(\cdot,0)}\nabla^j c_\chi\|_{0,\Omega_l}^2+e^{2s\tilde{d}_\ell}\right),\ i\in\mathbb{N}_n^*,\end{aligned}$$

according to (3.7). Summing up the above estimate over i in $\mathbb{N}_n^*$, it follows from (4.8) that

$$s\|e^{s\varphi(\cdot,0)}\nabla c_\chi\|_{0,\Omega_L}^2\leq C\left(\|e^{s\varphi(\cdot,0)}\nabla u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_L}^2+\sum_{j=0,1}\|e^{s\varphi(\cdot,0)}\nabla^j c_\chi\|_{0,\Omega_L}^2+e^{2s\tilde{d}_\ell}\right).$$

This and (4.10) yield the desired result. □

4.2. **Completion of the proof.** The proof is divided into three steps.
Step 1. The first step of the proof is to bound $u_\chi^{(2)}(\cdot,0)$ from above in the $e^{s\varphi(\cdot,0)}$-weighted $H^1(\Omega_L)$-norm topology, by the corresponding norms of $u_\chi^{(2)}$ and $u_\chi^{(3)}$ in Q_L, with the aid of the following technical result, borrowed from [3, Lemma 3.2]. Nevertheless, for the sake of completeness and for the convenience of the reader, we shall give the proof.

Lemma 4.2. *There exists a constant $s_*>0$ depending only on T such that we have*

$$\|z(\cdot,0)\|_{0,\Omega_L}^2\leq 2\left(s\|z\|_{0,Q_L}^2+s^{-1}\|\partial_t z\|_{0,Q_L}^2\right),$$

for all $s\geq s_*$ and $z\in H^1(-T,T;L^2(\Omega_L))$.

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for some $\mathfrak{p}_m>0$. Then for any $\mathfrak{p}_M\geq\max\{\|p_0\|_{C^0(\overline{D})},\|p_i\|_{C^1(\overline{D})},\ i\in\mathbb{N}_n^*\}$, there exist $s_*>0$ and $C>0$, depending only on $\mathfrak{p}_M$, such that the estimate

$$\|e^{s\varphi(\cdot,0)}v\|_{0,D}^2\leq Cs^{-1}\|e^{s\varphi(\cdot,0)}Pv\|_{0,D}^2,$$

holds for all $s\geq s_*$ and $v\in H_0^1(D)$.

Proof. Let η be defined by (3.22) for some fixed $\epsilon \in (0, T/2)$. Since

$$\begin{aligned}\|z(\cdot,0)\|_{0,\Omega_L}^2 &= \int_{-T}^0 \frac{d}{dt}\|\eta(t)z(\cdot,t)\|_{0,\Omega_L}^2 dt \\ &= 2\int_{-T}^0 \eta(t)^2 \Re\left(\int_{\Omega_L} z\overline{\partial_t z}(t,x)dx\right) + 2\int_{-T}^0 \eta\eta'(t)\|z(\cdot,t)\|_{0,\Omega_L}^2 dt,\end{aligned}$$

we infer from Young's inequality that

$$\|z(\cdot,0)\|_{0,\Omega_L}^2 \leq (s + 2\|\eta'\|_{L^\infty(\mathbb{R})})\|z\|_{0,Q_L}^2 + s^{-1}\|\partial_t z\|_{0,Q_L}^2,\ s>0,$$

The result follows from this upon taking $s^* = \|\eta'\|_{L^\infty(\mathbb{R})}/2$.

□

The proof, based on integration by parts and Young's inequality, can be found in [3, Lemma 3.2].

Namely, we apply Lemma 4.2 with $z = e^{s\varphi}\partial_i^j u_\chi^{(2)}$ for $i \in \mathbb{N}_n^*$ and $j = 0, 1$, getting

$$\|e^{s\varphi(\cdot,0)}\partial_i^j u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_L}^2 \leq C\left(s\|e^{s\varphi}\partial_i^j u_\chi^{(2)}\|_{0,Q_L}^2 + s^{-1}\|e^{s\varphi}\partial_i^j u_\chi^{(3)}\|_{0,Q_L}^2\right),\ s \geq s_*.$$

Summing up the above estimate over $i \in \mathbb{N}_n^*$ and $j = 0, 1$, we obtain for all $s \geq s_*$ that

$$\sum_{j=0,1}\|e^{s\varphi(\cdot,0)}\nabla^j u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_L}^2 \leq C\sum_{j=0,1}\left(s\|e^{s\varphi}\nabla^j u_\chi^{(2)}\|_{0,Q_L}^2 + s^{-1}\|e^{s\varphi}\nabla^j u_\chi^{(3)}\|_{0,Q_L}^2\right). \tag{4.11}$$

Step 2. The next step involves majorizing the right hand side of (4.11) with

$$\mathfrak{h}_k(s) := \sum_{j=0,1} s^{2(1-j)}\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j u_\chi^{(k)}\|_{0,\Sigma_L}^2,\ k = 2,3. \tag{4.12}$$

Indeed, since $u_\chi^{(k)}$, for $k = 2, 3$, is solution to (3.19) with $c = c_1$ and $f = f_{c_\chi}^{(k)} - g_k$, according to (4.4), then Corollary 3.2 yields

$$\begin{aligned}&s\sum_{j=0,1} s^{2(1-j)}\|\mathrm{e}^{s\varphi}\nabla_{x,t}^j u_\chi^{(k)}\|_{0,Q_L}^2 \\ \leq\ & C\left(\|\mathrm{e}^{s\varphi}f_{c_\chi}^{(k)}\|_{0,Q_L}^2 + \|\mathrm{e}^{s\varphi}g_k\|_{0,\tilde{Q}_\epsilon}^2 + s^3\mathrm{e}^{2s\tilde{d}_\ell}\|u_\chi^{(k)}\|_{1,Q_L}^2 + s\mathfrak{h}_k(s)\right),\end{aligned}$$

for s large enough. In light of (4.11) this entails that

$$\begin{aligned}&\sum_{j=0,1}\|e^{s\varphi(\cdot,0)}\nabla^j u_\chi^{(2)}(\cdot,0)\|_{0,\Omega_L}^2 \\ \leq\ & C\sum_{k=2,3}\left(\|\mathrm{e}^{s\varphi}f_{c_\chi}^{(k)}\|_{0,Q_L}^2 + \|\mathrm{e}^{s\varphi}g_k\|_{0,\tilde{Q}_\epsilon}^2 + s^3\mathrm{e}^{2s\tilde{d}_\ell}\|u_\chi^{(k)}\|_{1,Q_L}^2 + s\mathfrak{h}_k(s)\right)\end{aligned} \tag{4.13}$$

Further, recalling (4.5), we see from (1.5) and (1.12) (resp., from (1.5)-(1.6), (1.12) and (3.7)) that the first (resp., second) term of the sum in the right hand side of (4.13) is upper bounded up to some multiplicative constant, by $\sum_{j=0,1}\|\mathrm{e}^{s\varphi}\nabla^j c_\chi\|_{0,Q_L}^2$ (resp., $\mathrm{e}^{2s\tilde{d}_\ell}(\|u^{(k)}\|_{1,Q_L}^2 + 1)$). From this and Lemma 4.1 then follows for s sufficiently large that

$$\begin{aligned}&C\,s\sum_{j=0,1}\|e^{s\varphi(\cdot,0)}\nabla^j c_\chi\|_{0,\Omega_L}^2 \\ \leq\ & \sum_{j=0,1}\|\mathrm{e}^{s\varphi}\nabla^j c_\chi\|_{0,Q_L}^2 + \mathrm{e}^{2s\tilde{d}_\ell} + \sum_{k=2,3}\left(s^3\mathrm{e}^{2s\tilde{d}_\ell}\|u^{(k)}\|_{1,Q_L}^2 + s\mathfrak{h}_k(s)\right).\end{aligned} \tag{4.14}$$

Step 3. Finally, we notice from (3.3) that

$$\|e^{s\varphi}\nabla^j c_\chi\|_{0,Q_L} = \|\rho_s^{1/2} e^{s\varphi(\cdot,0)}\nabla^j c_\chi\|_{0,\Omega_L},\ j=0,1, \tag{4.15}$$

where $\rho_s(x) := \int_{-T}^{T} e^{2s(\varphi(x,t)-\varphi(x,0))}dt = \int_{-T}^{T} e^{-2s\varphi(x,0)(1-\exp(-\gamma t^2))}dt$ for all $x \in \Omega_L$. Bearing in mind that $\varphi(x,0) \geq \widetilde{\kappa} := e^{\gamma(\inf_{x'\in\omega}|x'-\delta a'|^2-L^2)} > 0$ for all $x \in \Omega_L$, we get that

$$\|\rho_s\|_{L^\infty(\Omega_L)} \leq \int_{-T}^{T} e^{-2s\widetilde{\kappa}(1-\exp(-\gamma t^2))}dt,\ s>0.$$

Therefore we have $\lim_{s\to+\infty}\rho_s = 0$, uniformly in Ω_L, by the dominated convergence theorem, so we derive from (4.14)-(4.15) that

$$s\sum_{j=0,1}\|e^{s\varphi(\cdot,0)}\nabla^j c_\chi\|_{0,\Omega_L}^2 \leq C\left(e^{2s\tilde{d}_\ell} + \sum_{k=2,3}\left(s^3 e^{2s\tilde{d}_\ell}\|u^{(k)}\|_{1,Q_L}^2 + s\mathfrak{h}_k(s)\right)\right), \tag{4.16}$$

upon taking s sufficiently large. With reference to (3.5)–(3.7), this entails that

$$\sum_{j=0,1}\|\nabla^j c_\chi\|_{0,\Omega_\ell}^2 \leq C\sum_{k=2,3}\left(s^2 e^{-2s(d_\ell-\tilde{d}_\ell)} + \mathfrak{h}_k(s)\right). \tag{4.17}$$

Here we used (1.5)-(1.12) and the embedding $\Omega_\ell \subseteq \Omega_L$ in order to substitute Ω_ℓ for Ω_L in the left hand side of (4.16). Now, taking into account that $\tilde{d}_\ell < d_\ell$ and noting the second equality in (4.4) and (4.12), we end up getting the desired result from (4.17).

Appendix A. **.** In this appendix we prove Theorem 2.1. To this end, we refer to the IBVP (2.1) and start by recalling from [2][19, Sect. 3, Theorem 8.2] the following existence and uniqueness result.

Proposition A.1. *Assume that $f \in C^0([0,T];D(A^0))$, $g \in D(A^{1/2})$ and $h \in D(A^0)$. Then there exists a unique solution v to* (2.1) *such that*

$$\partial_t^k v \in C^0([0,T];D(A^{(1-k)/2})),\ k=0,1. \tag{A.1}$$

Moreover we have the estimate

$$\sum_{k=0}^{1}\sup_{t\in[0,T]}\|\partial_t^k v(\cdot,t)\|_{D(A^{(1-k)/2})} \leq C\left(\|f\|_{C^0([0,T];D(A^0))} + \|g\|_{D(A^{1/2})} + \|h\|_{D(A^0)}\right). \tag{A.2}$$

A.1. **Improved regularity.**

Proposition A.2. *Assume that $f \in C^1([0,T];D(A^0))$, $g \in D(A)$ and $h \in D(A^{1/2})$. Then the solution v to* (2.1) *satisfies*

$$\partial_t^k v \in C^0([0,T];D(A^{(2-k)/2})),\ k=0,1,2, \tag{A.3}$$

[2]Upon taking $H := D(A^0) = L^2(\Omega)$, $V := D(A^{1/2}) = H_0^1(\Omega)$ and $a(t;u,v) := \int_\Omega c(x)\nabla u(x)\cdot\overline{\nabla v(x)}dx$ for $u,v \in V$ and all $t \in [0,T]$, in the definitions (8.2)-(8.3). Evidently, since the system under study is autonomous here, the sesquilinear form a is time-independent, and hence continuously differentiable with respect to t.

and we have the estimate

$$\sum_{k=0}^{2} \sup_{t\in[0,T]} \|\partial_t^k v(\cdot,t)\|_{D(A^{(2-k)/2})} \leq C\left(\|f\|_{C^1([0,T];D(A^0))} + \|g\|_{D(A)} + \|h\|_{D(A^{1/2})}\right). \tag{A.4}$$

Proof. By differentiating (2.1) with respect to t, we check that $w := \partial_t v$ obeys

$$\begin{cases} \partial_t^2 w + Aw = \partial_t f & \text{in } Q \\ w(\cdot,0) = h,\ \partial_t w(\cdot,0) = f(\cdot,0) - Ag & \text{in } \Omega. \end{cases} \tag{A.5}$$

Since $f(\cdot,0)-Ag$ is lying in $D(A^0)$ then we have $\partial_t^{k+1} v = \partial_t^k w \in C^0([0,T];D(A^{(1-k)/2}))$ for $k=0,1$, with the estimate

$$\begin{aligned} &\sum_{k=0}^{1} \sup_{t\in[0,T]} \|\partial_t^{k+1} v(\cdot,t)\|_{D(A^{(1-k)/2})} \\ \leq\ & C\left(\|\partial_t f\|_{C^0([0,T];D(A^0))} + \|h\|_{D(A^{1/2})} + \|f(\cdot,0) - Ag\|_{D(A^0)}\right) \\ \leq\ & C\left(\|f\|_{C^1([0,T];D(A^0))} + \|g\|_{D(A)} + \|h\|_{D(A^{1/2})}\right) \end{aligned} \tag{A.6}$$

by Proposition A.1. Further, as $Av = f - \partial_t^2 v$ from the first line in (2.1), we get that $v \in C^0([0,T];D(A))$, and that $\|v(\cdot,t)\|_{D(A)}$ is majorized by the right hand side of (A.6), uniformly in $t\in[0,T]$. This and (A.6) yield the desired result. □

Armed with Proposition A.2 we may now prove the statement of Theorem 2.1, claiming higher regularity for the solution to (2.1).

A.2. **Higher regularity: Proof of Theorem 2.1.** The proof is by an induction on $m \in \mathbb{N}$.
a) The case $m = 0$ follows from Proposition A.1.
b) We assume that the theorem is valid for some $m \in \mathbb{N}$ and suppose that

$$\begin{cases} g \in D(A^{(m+2)/2}),\ h \in D(A^{(m+1)/2}), \\ \partial_t^k f \in C^0([0,T];D(A^{(m+1-k)/2})),\ k = 0,\ldots,m+1. \end{cases} \tag{A.7}$$

We use the same strategy as in the proof of Proposition A.2. That is we differentiate (2.1) with respect to t and get that $w := \partial_t v$ is solution to (A.5). Next, using that $h \in D(A^{(m+1)/2})$, $f(0) - Ag \in D(A^{m/2})$ and

$$\partial_t^k(\partial_t f) = \partial_t^{k+1} f \in C^0([0,T];D(A^{(m-k)/2})),\ k = 0,\ldots,m,$$

from (A.7), we get that $\partial_t^{k+1} v = \partial_t^k w \in C^0([0,T];D(A^{(m+1-k)/2}))$ for $k = 0,1,\ldots,m+1$, and the estimate:

$$\begin{aligned} &\sum_{k=0}^{m+1} \sup_{t\in[0,T]} \|\partial_t^{k+1} v(\cdot,t)\|_{D(A^{(m+1-k)/2})} \\ \leq\ & C\left(\sum_{k=0}^{m} \|\partial_t^{k+1} f\|_{C^0([0,T];D(A^{(m-k)/2}))} + \|h\|_{D(A^{(m+1)/2})} + \|f(0) - Ag\|_{D(A^{m/2})}\right). \end{aligned}$$

This entails $\partial_t^k v \in C^0([0,T]; D(A^{(m+2-k)/2}))$ for $k = 1, \ldots, m+2$, and

$$\begin{aligned} & C \sum_{k=1}^{m+2} \sup_{t \in [0,T]} \|\partial_t^k v(\cdot,t)\|_{D(A^{(m+2-k)/2})} \\ \leq & \sum_{k=0}^{m+1} \|\partial_t^k f\|_{C^0([0,T];D(A^{(m+1-k)/2}))} + \|h\|_{D(A^{(m+1)/2})} + \|g\|_{D(A^{(m+2)/2})}. \end{aligned} \quad \text{(A.8)}$$

Further, as $Av = f - \partial_t^2 v$ from the first line in (2.1), we find out that

$$\begin{aligned} C\|v(\cdot,t)\|_{D(A^{(m+2)/2})} & \leq \|Av(\cdot,t)\|_{D(A^{m/2})} + \|v\|_{D(A^0)} \\ & \leq \|f(\cdot,t)\|_{D(A^{m/2})} + \|\partial_t^2 v(\cdot,t)\|_{D(A^{m/2})} + \|v\|_{D(A^0)}. \end{aligned} \quad \text{(A.9)}$$

Here we used the identity

$$\|v(\cdot,t)\|^2_{D(A^{(m+2)/2})} = \|Av(\cdot,t)\|^2_{D(A^{m/2})} + \sum_{k=0}^{1} \|A^{k/2} v(\cdot,t)\|^2_{D(A^0)},$$

and the estimate $\|A^{1/2} v(\cdot,t)\|_{D(A^0)} \leq \sum_{k=0}^{1} \|A^k v(\cdot,t)\|_{D(A^0)}$. Since $\|v\|_{D(A^0)}$ and $\|\partial_t^2 v(\cdot,t)\|_{D(A^{m/2})}$ are majorized by the right hand side of (A.8), uniformly in $t \in [0,T]$, (A.8)-(A.9) yield the assertion of the theorem for $m+1$.

This terminates the proof of Theorem 2.1.

Appendix B. **.** In this second appendix we prove Proposition 2.3 with the help of the following elliptic boundary regularity result.

B.1. **Elliptic boundary regularity.** In this subsection we extend the classical elliptic boundary regularity result for the operator $\nabla \cdot c \nabla$, which is well known in any sufficiently smooth bounded subdomain of $\mathbb{R}^n$ (see e.g. [10, Sect. 6.3, Theorem 5]), to the case of the infinite waveguide Ω under study. The proof of this result boils down to [16, Lemma 2.4] which claims elliptic boundary regularity for the Dirichlet Laplacian in Ω.

Lemma B.1. *Let r be a nonnegative integer. We assume that $\partial\omega$ is C^{r+2} and that $c \in W^{r+1,\infty}(\Omega)$ obeys (1.2). Then, for any $\varphi \in H^r(\Omega)$, there exists a unique solution $v \in H^{r+2}(\Omega)$ to the boundary problem*

$$\begin{cases} -\nabla \cdot c(x) \nabla v = \varphi & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad \text{(B.1)}$$

Moreover we have the estimate

$$\|v\|_{r+2,\Omega} \leq C_r \|\varphi\|_{r,\Omega}, \quad \text{(B.2)}$$

where C_r is a positive constant depending only on r, ω, the constant c_m appearing in (1.2) and $\|c\|_{W^{r+1,\infty}(\Omega)}$.

Proof. The proof is by induction on r.
a) We first consider the case $r = 0$. Due to (1.2) there is a unique solution $v \in H_0^1(\Omega)$ to (B.1) by the Lax-Milgram theorem. Moreover v satisfies the energy estimate

$$\|v\|_{1,\Omega} \leq C \|\varphi\|_{0,\Omega}, \quad \text{(B.3)}$$

where the constant $C > 0$ depends only on ω and c_m. Here we used (1.2) and the Poincaré inequality, which holds true in Ω since ω is bounded. Furthermore, v is solution to the boundary value problem

$$\begin{cases} -\Delta v = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.4}$$

where

$$f := c^{-1} \left(\varphi + \nabla c \cdot \nabla v \right). \tag{B.5}$$

Since $f \in L^2(\Omega)$ then $v \in H^2(\Omega)$ by [16, Lemma 2.4], and $\|v\|_{2,\Omega}$ is upper bounded, up to some multiplicative constant depending only on ω, by $\|f\|_{0,\Omega}$. As a consequence we have

$$\|v\|_{2,\Omega} \leq C' \left(\|\varphi\|_{0,\Omega} + \|v\|_{1,\Omega} \right),$$

from (B.5), the constant $C' > 0$ depending only on ω, c_m and $\|c\|_{W^{1,\infty}(\Omega)}$. This and (B.3) yield (B.2) with $r = 0$.

b) Suppose that the statement of the lemma is true for $r \in \mathbb{N}$ fixed, and assume that $\partial\omega$ is C^{r+3}, $c \in W^{r+2,\infty}(\Omega)$ and $\varphi \in H^{r+1}(\Omega)$. Hence the solution v to (B.4) belongs to $H^{r+2}(\Omega)$ and satisfies the estimate (B.2), by induction assumption, and we have $f \in H^{r+1}(\Omega)$ in virtue of (B.5). Further, v being solution to (B.4) where the boundary $\partial\omega$ is C^{r+3} then $v \in H^{r+3}(\Omega)$ by [16, Lemma 2.4]. Moreover $\|v\|_{r+3,\Omega}$ is upper bounded (up to some multiplicative constant depending only on r and ω) by $\|f\|_{r+1,\Omega}$. From this and (B.5) then follows that

$$\|v\|_{r+3,\Omega} \leq C'' \left(\|\varphi\|_{r+1,\Omega} + \|v\|_{r+2,\Omega} \right),$$

where the constant $C'' > 0$ depends only on r, ω, c_m and $\|c\|_{W^{r+2,\infty}(\Omega)}$. Putting this together with (B.2), we obtain (B.2) where r is replaced by $r+1$, proving that the statement of the lemma remains valid upon substituting $r+1$ for r. □

B.2. **Proof of Proposition 2.3.** It suffices to show that

$$D(A^k) \subset \{u \in H^{2k}(\Omega),\ u, Au,\ \dots, A^{m-1}u \in H_0^1(\Omega)\}, \tag{B.6}$$

and

$$\|u\|_{2k,\Omega} \leq c(k) \|u\|_{D(A^k)},\ u \in D(A^k). \tag{B.7}$$

The proof is by induction on m.

a) We start with $m = 1$ and notice from the very definition of $A^{1/2}$ that $D(A^{1/2}) = D(q_A) = H_0^1(\Omega)$. Moreover we have

$$\|A^{1/2}u\|_{0,\Omega}^2 = q_A[u] \geq c_m \|\nabla u\|_{0,\Omega}^2,\ u \in D(A^{1/2}),$$

in virtue of (1.2). Bearing in mind that $c_m \in (0,1)$, we obtain that $\|u\|_{1,\Omega} \leq c_m^{-1/2} \|u\|_{D(A^{1/2})}$ for every $u \in D(A^{1/2})$. This establishes (B.7) for $k = 1/2$.

Similarly, bearing in mind that $D(A) = \{u \in H_0^1(\Omega),\ Au \in L^2(\Omega)\}$, we apply Lemma B.1 with $r = 0$ and $\varphi = Au$, where $u \in D(A)$ is arbitrary. We find that $u \in H^2(\Omega)$ satisfies $\|u\|_{2,\Omega} \leq C_0 \|Au\|_{0,\Omega}$, which entails (B.6)-(B.7) for $k = 1$.

b) Let us now suppose that the statement of the lemma is true for some $m \in \mathbb{N}^*$ fixed. Pick $k \in \{m - 1/2, m\}$ and assume that $\partial\omega$ is $C^{2(k+1)}$ and that $c \in W^{2k+1,\infty}(\Omega)$ satisfies (1.2). As $D(A^{k+1}) = \{u \in D(A^k),\ Au \in D(A^k)\}$, we deduce from the induction assumption that we have

$$D(A^{k+1}) = \{u \in H^{2k}(\Omega),\ Au \in H^{2k}(\Omega) \text{ and } u, \dots, A^m u \in H_0^1(\Omega)\},$$

with $\|A^j u\|_{2k,\Omega} \leq c(k)\|A^j u\|_{D(A^k)}$ for $j = 0, 1$. Thus, applying Lemma B.1, with $r = 2k$ and $\varphi = Au$, for $u \in D(A^{k+1})$, we get that $u \in H^{2(k+1)}(\Omega)$, proving (B.6) where $(k+1, m+1)$ is substituted for (k, m). Moreover, it holds true that

$$\|u\|_{2(k+1),\Omega} \leq C_{2k}\|Au\|_{2k,\Omega},$$

and since $Au \in D(A^k)$, the induction assumption yields $\|Au\|_{2k,\Omega} \leq c(k)\|Au\|_{D(A^k)}$. Therefore $\|u\|_{2(k+1),\Omega}$ is majorized, up to a multiplicative constant depending only on ω, c and m, by $\|u\|_{D(A^{k+1})}$, which is (B.7) where $k+1$ is substituted for k.

This completes the proof of Proposition 2.3.

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